Multilevel Marketing: Pyramid-Shaped Schemes or Exploitative Scams?*

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Abstract

We identify the conditions on the tendency of agents to spread information by word of mouth, under which a principal can design a pyramid scam to exploit a network of boundedly rational agents whose beliefs are coarse. Our main result is that a pyramid scam is sustainable only if its underlying reward scheme compensates the participants based on multiple levels of their downlines (e.g., for recruiting new members to the pyramid and for recruitments made by these new members). Motivated by the growing discussion on the legitimacy of multilevel marketing schemes and their resemblance to pyramid scams, we use our model to compare the two phenomena based on their underlying compensation structure.

Keywords: pyramid scams; multilevel marketing; analogy-based expectations; coarse feedback; bounded rationality.

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1 Introduction

What delineates pyramid scams from legitimate multilevel marketing enterprises? Dramatic recent growth[1] in the multilevel marketing (MLM) industry—which over the past five years has engaged over 20 million[2] Americans—has raised the urgency of this question for consumer protection agencies. MLM companies such as Avon, Amway, Herbalife, and Tupperware use independent representatives to sell their products to friends and acquaintances. They all promote the opportunity of starting one's own business and making extra income; however, some (see, e.g., Bort, 2016) view these companies as pyramid scams whose main purpose is to take advantage of vulnerable individuals.

The MLM industry's questionable legitimacy and its resemblance to fraudulent pyramid scams received considerable attention in the mainstream media[3] following a recent feud between Herbalife and the hedge-fund tycoon Bill Ackman, a dispute that led to an FTC investigation against the former party (FTC, 2016a). Identifying whether a particular company is a legitimate one, or an exploitative pyramid scam that promotes useless goods and services in order to disguise itself as a legitimate firm, can be a daunting task. One obstacle is that MLM companies typically sell products whose quality is difficult to assess, such as vitamins and nutritional supplements. The common wisdom among practitioners is that a company is legitimate if the distributors are encouraged to sell the product, and it is an illegal pyramid if it prioritizes recruitment over selling (FTC, 2016b). However, it is extremely difficult to determine the company's true "selling point" and, in practice, it is challenging to distinguish between sales to members and sales to the general public.

The objective of this paper is to draw the boundary between the two phenomena based on their underlying compensation schemes. The premise of our analysis is that the potential distributors are strategic, and that the MLM company (or the pyramid organizer) chooses a compensation scheme while taking these prospective distributors' incentives into account. To illustrate the structure of the potential reward schemes, consider the following example.

Example 1 *The reward scheme R pays every distributor a commission of b_1 for each retail sale that he makes and a commission of a_1 for every agent whom he recruits*

[1] Membership in MLMs is substantial and growing. For instance, according to a recent New York Times article, Amway has 1.9 million distributors in China alone (McMorrow and Myers, 2018).

[2] According to the Direct Selling Association's (DSA) annual report (DSA, 2016).

[3] See, e.g., Wieczner, 2017; McCrum, 2014, 2016; McKown, 2017; Multi-level Marketing in America, 2015; Moyer, 2018; Parloff, 2015, 2016; Pierson, 2017; Suddath, 2018; Truswell, 2018.

to the sales force. The scheme R' pays each distributor a commission of b'_1 for each retail sale that he makes and a commission of b'_2 for each retail sale made by one of his recruits. It also pays each distributor a'_1 for each of his recruits and a'_2 for each of his recruits' recruits. Both schemes charge a license fee[4] *of $\phi \geq 0$ from each distributor. We refer to a_1, a'_1, and a'_2 as recruitment commissions, and to reward schemes such as R (respectively, R') as one-level (respectively, multilevel) schemes as they compensate the distributors based on the first level (respectively, multiple levels) of their downline.*

Observe that both R and R' compensate the distributors for recruiting others to work for the company. However, in practice, the bulk of the MLM industry uses multilevel schemes (DSA, 2014) such as R', rather than one-level schemes such as R. Moreover, even though there is no obvious reason why one-level schemes such as R cannot be used for the purpose of sustaining a pyramid scam, various companies that were deemed[5] pyramid scams used multilevel reward schemes. What can explain these stylized facts? Can a legitimate company benefit from charging entry fees, paying recruitment commissions, or offering multiple routes through which individuals can join the sales force? Does the answer depend on whether the company promotes genuine goods or just the opportunity to recruit others to the sales force?

In order to address the above questions, we develop a model of word-of-mouth marketing in which a scheme organizer (SO) tries to sell a good to a network of agents that is formed randomly and sequentially. In order to reach larger parts of the network, the SO sells distribution licenses to some of the agents. Distributors can sell units of the good as well as distribution licenses, and they are compensated according to a reward scheme that is chosen in advance by the SO. A key feature of the model is that each agent's likelihood of meeting new entrants (i.e., potential buyers and distributors) goes down as time progresses, which makes this "business opportunity" unattractive to agents who receive an offer to join the sales force late in the game, when there are fewer opportunities to sell the good and recruit others to the sales force.

Assume for a moment that the good has no intrinsic value such that the only "products" that are being traded are distribution licenses. If there exists a reward scheme such that the SO makes a strictly positive expected profit in its induced game, then we have a *pyramid scam.* Pyramid scams are, roughly speaking, zero-sum games and, therefore, fully rational economic agents will never participate in them. Nevertheless, we observe countless such scams in practice (see, e.g., Keep and Vander Nat, 2014,

[4]In practice, such fees are often presented as training costs or a requirement to purchase initial stock.

[5]See, e.g., BurnLounge (FTC, 2014b) and Fortune High-Tech Marketing (FTC, 2014a).

and the references therein). Hence, if we wish to better understand such scams and their underlying compensation schemes, we must depart from the classic rational expectations framework. We shall use Jehiel's (2005) elegant framework of *analogy-based expectations equilibrium* to relax the requirement that the agents have a perfect understanding of the other agents' behavior in every possible contingency, while maintaining that the agents' beliefs are *statistically correct*.[6]

Under the behavioral model, each agent correctly predicts (and best responds to) the other agents' *average behavior*. However, the agents neglect the fact that the other agents' strategies might be time-contingent. This mistake leads them to mispredict the other agents' "marginal" equilibrium behavior (e.g., to think that it might be possible to recruit new participants to the pyramid late in the game). Despite these mistakes, the agents' beliefs are statistically correct and can be interpreted as resulting from the use of a simplified model of the other players' behavior, or as learning from partial feedback about the behavior in similar past interactions (e.g., the sales force's average performance in past schemes organized by the SO).

An individual who contemplates joining a pyramid often tries to *assess his ability to recruit* others as well as his potential recruits' respective ability. A general insight that emerges from the model is that individuals who understand others' average behavior *do not overestimate their own ability to recruit by much*, if at all. Therefore, the SO cannot exploit such individuals and sustain a pyramid scam by means of one-level schemes such as R. Multilevel schemes such as R' introduce additional variables for the prospective participants to mispredict (e.g., their recruits' ability to recruit). All of these prediction mistakes are small. However, the *accumulation of these small prediction errors* enables the SO to sustain a pyramid scam.

We provide necessary and sufficient conditions—on the number of agents and their tendency to spread information by word of mouth—under which the SO can sustain a pyramid scam, and we show that multilevel schemes can support such a scam whereas one-level schemes cannot generate a strictly positive expected profit for the SO.

In order to better understand legitimate MLM companies, we shall examine a setting in which the goods have an intrinsic value such that the SO can benefit from selling them. The reward scheme's objective in this case is to incentivize the distributors both to sell the products and to propagate information about them, while maintaining a low overhead. We solve for the SO's optimal scheme under two behavioral assumptions. First, we show that if the agents are fully rational, then schemes that maximize the SO's expected equilibrium profit do not charge license fees, nor do they pay recruitment

[6] In Section 5, we shall discuss the implications of other behavioral models for our results.

commissions. Second, when the SO faces a population of analogy-based reasoners, then if the number of agents is sufficiently large, under mild assumptions, optimal schemes compensate the distributors only for sales and do not charge license fees. Thus, the two pyramidal components—recruitment commissions and license fees—are not in use when the good is intrinsically valued even though the agents are vulnerable to deceptive practices.

The main contributions of the paper are fourfold. First, we develop a model that enables us to better understand what makes pyramid scams work. Second, our results suggest an explanation as to why such scams often rely on multilevel reward schemes and dubious "passive income" promises (see Securities and Exchange Commission, 2013). Third, our analysis shows how the presence of rational agents can protect vulnerable agents from participating in exploitative pyramid scams in noncompetitive environments. Finally, our analysis of legitimate MLM enterprises allows us to more transparently draw the dividing line between exploitative scams and legitimate MLM companies.

Related literature

We use analogy-based expectations equilibrium, which was developed in Jehiel (2005) and extended in Jehiel and Koessler (2008), as our behavioral framework. A closely related concept, "cursed equilibrium," was developed by Eyster and Rabin (2005) for games of incomplete information. In a cursed equilibrium, agents fail to realize the extent to which the other players' actions depend on their private information. Piccione and Rubinstein (2003) study intertemporal pricing when consumers reason in terms of a coarse representation of the correct equilibrium price distribution. Other prominent models in which players reason in terms of a coarse representation of the world are Mullainathan et al. (2008), Jehiel (2011), Eyster and Piccione (2013), Guarino and Jehiel (2013), and Steiner and Stewart (2015). In Eyster and Rabin (2010), in the context of social learning, agents best respond to a belief that their predecessors are cursed (i.e., do not learn from their own predecessors' behavior).

Our work relates to a strand of the behavioral industrial organization literature in which rational firms exploit boundedly rational agents. Spiegler (2011) offers a textbook treatment of such models. In Eliaz and Spiegler (2006, 2008), a principal interacts with agents who differ in their ability to predict their future tastes. Grubb (2009) studies contracting when agents are overconfident about the accuracy of their forecasts of their own future demand. In Laibson and Gabaix (2006), firms may hide information about add-on prices from unaware consumers. Heidhues and Kőszegi (2010) study exploitative credit contracts when consumers are time-inconsistent. In the context of auctions, Crawford et al. (2009) show that agents who are characterized by level-k thinking can be exploited by a rational auctioneer. Eliaz and Spiegler (2007, 2008, 2009) apply a mechanism-design approach to speculative trade between agents who hold different prior beliefs.

This article is also related to the "Dutch Books" literature that studies the vulnerability of non-standard preferences (see, e.g., Yaari, 1985) or of non-standard decision-making procedures to exploitative transactions. For example, Rubinstein and Spiegler (2008) examine the extent to which agents who employ a sampling procedure in the spirit of the S-1 equilibrium (Osborne and Rubinstein, 1998) are vulnerable to exploitative transactions offered by a rational market maker. Laibson and Yariv (2007) show that competitive markets may protect agents with non-standard preferences from exploitative schemes.

Pyramid scams are closely related to speculative bubbles (see Brunnermeier and Oehmke, 2013, for a recent survey of the bubbles literature). Tirole (1982) shows that speculative bubbles cannot exist under the rational expectations model if it is commonly known that there are no potential gains from trade. In Abreu and Brunnermeier's (2003) seminal work (which we shall discuss in detail in the concluding section) an exogenous process generates a finite horizon bubble. Rational traders become aware of the bubble sequentially and try to time the market (i.e., ride the bubble) rather than to attack the bubble immediately. In DeLong et al. (1990), rational speculators anticipate that positive-feedback noise traders will push an asset's price above its fundamental value in the next period, and therefore purchase the asset in order to resell it at an inflated price in the future. These purchases "fuel" the noise traders' demand, exacerbate the mis-pricing, and increase the speculators' profit. Bianchi and Jehiel (2010) show that the analogy-based expectations equilibrium logic can sustain both bubbles and crashes in equilibrium. Harrison and Kreps (1978) and Scheinkman and Xiong (2003) show that heterogeneous prior beliefs lead to overpricing when there are short-sale constraints.

Pyramids and multilevel marketing schemes have received some attention outside of the economics literature. A strand of the computer science literature (e.g., Emek et al., 2011; Drucker and Fleischer, 2012) focuses on multilevel marketing mechanisms' robustness to Sybil attacks. The marketing literature has addressed ethical issues in multilevel marketing and the resemblance of such schemes to pyramid scams (a comprehensive overview is provided in Keep and Vander Nat, 2014). The common view in that literature is that a company is a pyramid scam if the participants' compensation is based primarily on recruitment rather than retail sales to end users (see, e.g., Koehn, 2001; Keep and Vander Nat, 2002). Gastwirth (1977) and Bhattacharya and Gastwirth (1984) use the random recursive tree model to examine two real-world scams and demonstrate that only a small fraction of the participants can cover the entry fees. In none of the above-mentioned models, however, is there strategic interaction.

The paper proceeds as follows. We present the model in Section 2 and analyze pure pyramid scams in Section 3. Section 4 examines legitimate MLM companies. Section 5 studies a model of non-common priors and Section 6 concludes by discussing several extensions of the model. In Appendix A, in order to demonstrate the robustness of our findings, we present a semistationary modification of the model in which there is uncertainty about the length of the game. All proofs are relegated to Appendix B.

2 The Model

There is a scheme organizer (SO) who produces a good free of cost and with no capacity constraints, and a set of agents $I = \{1, ..., n\}$. Each agent $i \in I$ is characterized by a unit demand and two numbers: his willingness to pay $\omega_i \in \{0, 1\}$ and his talkativeness $\psi_i \in \{0, 1\}$. The term "talkativeness" will be clarified soon. For every agent $i \in I$, we assume that ω_i and ψ_i are drawn independently and we denote $p := Pr(\psi_i = 1)$ and $q := Pr(\omega_i = 1)$, respectively.

In order to purchase the good, an agent must first become aware of its existence. The agents learn about its existence when they interact with other players.

Meetings, distribution, and word of mouth

Time $t = 0, 1, 2, ..., n$ is discrete. In period 0, the SO enters the game. In each period $t \geq 1$, nature draws a new agent (uniformly at random) who enters the game and meets one player who is chosen uniformly at random out of the t players who entered the game prior to period t. For example, the second entrant meets either the SO or the first entrant, each with probability $\frac{1}{2}$. We often use i_t to denote the t-th entrant. Let G (respectively, G^t) denote the directed tree, rooted at the SO, that results at the end of this process (respectively, at the end of period t) and let G_i (respectively G_i^t) denote the subtree of G (respectively, G^t) rooted at $i \in I$. For each tree G, we denote the number of nodes in G by $|G|$, and the length of the path between $i \in I \cup \{SO\}$ and $j \in I$ by $d_G(i, j)$.

How does an agent become a distributor? When a new agent $i \in I$ enters the game, if he meets an agent who is already a distributor (respectively, the SO), then the latter

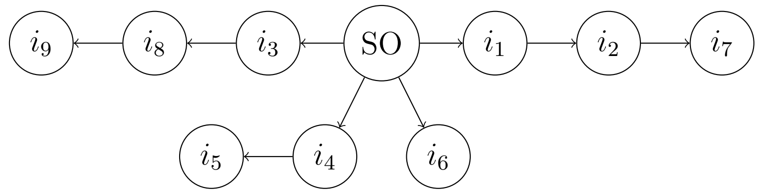

Figure 1: A snapshot of G at the end of period 9 (i.e., G^9).

can make i an offer to become a distributor. If the entrant meets an agent who is not a distributor, then he can still receive such an offer from the nearest distributor if there is a path of talkative agents connecting that distributor to the entrant (i.e., if $\psi_j = 1$ for every agent j who is on that path).

Formally, for each $t \geq 1$, let D_t be a set that includes the SO and the distributors at the beginning of period t. Recall that the t-th entrant is equally likely to meet each of the players who entered the game previously. For each $j \in D_t$, if there is a directed path connecting j to the t-th entrant such that every agent on the path is (i) talkative and (ii) not a distributor (i.e., if $\psi_l = 1$ and $l \notin D_t$ for every agent l on that path), then, in period t, j can offer the t-th entrant the opportunity to become a distributor. If the latter receives such an offer, he can accept it and become a distributor or reject it. In addition, regardless of whether an offer is made by j, the t-th entrant purchases a unit of the good for personal consumption from j at a price η^R that is predetermined by the SO if and only if[7] $\omega_{i_t} \geq \eta^R$.

Observe that if there is an agent $i \notin D_t$ on the path connecting the SO to the t-th entrant and $\psi_i = 0$, then the t-th entrant does not receive an offer to become a distributor and does not purchase the good.

To illustrate the game, consider a history such that G^9 is as presented in Figure 1, $D_{10} = \{SO, i_1, i_2, i_3\}$, $\psi_{i_4} = 0$, and $\psi_{i_5} = \psi_{i_6} = \psi_{i_7} = \psi_{i_8} = \psi_{i_9} = 1$ (i.e, i_4 is not talkative and i_5,i_6,i_7,i_8, and i_9 are talkative). Recall that i_{10} is equally likely to meet each of the 10 players who entered the game prior to period 10. If i_{10} meets a player $j \in D_{10}$, then j decides whether or not to make i_{10} an offer to become a distributor. If i_{10} meets a non-distributor $j \in \{i_4, i_5\}$, then i_{10} will not receive an offer or purchase the good as the path from the SO is "blocked" by agent i_4 who is not talkative and not a distributor. Note that agent i_{10} cannot hear about the good/scheme from i_5 as i_5 himself did not learn about it when he met i_4. However, if i_{10} meets a non-distributor $j \in \{i_6, i_7, i_8, i_9\}$, then j will tell i_{10} about the good/scheme and refer him to the nearest member of D_{10} who will decide whether or not to make an offer to i_{10}. Thus, i_7 will refer i_{10} to i_2, i_6 will refer him to the SO, and i_9 will refer him to i_3 via i_8.

[7] It is possible to make the price and the decision to sell/buy the good endogenous without changing the main results in this article.

Reward schemes, payoffs, and information

The distributors receive commissions according to a *reward scheme* that is chosen in advance by the SO. Each reward scheme R includes four components:

- An entry fee $\phi^R \geq 0$.
- Recruitment commissions: $a_1^R, a_2^R, a_3^R, ... \geq 0$.
- Sales commissions: $b_1^R, b_2^R, b_3^R, ... \geq 0$.
- A price $\eta^R \geq 0$ at which each unit of the good is sold.

When the t-th entrant purchases a unit of the good (respectively, becomes a distributor), each distributor $l \in D_t$ who is on the path connecting the SO to the t-th entrant obtains a commission of b_{y+1}^R (respectively, a_{y+1}^R) from the SO, where $y \geq 0$ is the number of distributors on the path connecting l to the t-th entrant. In addition, the SO receives η^R (respectively, ϕ^R) from the t-th entrant.

We assume that each agent who becomes a distributor incurs a cost of $c \geq 0$ that reflects learning about the good and how to sell it. When an agent contemplates purchasing a distribution license (i.e., becoming a distributor), he weighs the expected sum of commissions that he will obtain (given R and his beliefs about the other players' behavior) against the total cost of becoming a distributor $c + \phi^R$.

We shall restrict our attention to schemes for which $a_\tau^R \leq \phi^R$ and $b_\tau^R \leq \eta^R$ for each $\tau \geq 1$, and refer to such schemes as *incentive-compatible* (IC) schemes. Note that non-IC schemes provide each distributor with an incentive to misreport to the SO that he sold additional distribution licenses or units of the good.[8] For example, consider a scheme R, such that $a_1^R > \phi^R$, and a distributor i. By misreporting that he sold a license to some fake identity j, distributor i will be eligible for a commission of a_1^R from the SO. This misreporting is beneficial only if $a_1^R > \phi^R$ as i will have to pay j's entry fee.

Each scheme R induces a game of perfect information, which we shall denote by $\Gamma(R)$. The SO's highest equilibrium expected profit in $\Gamma(R)$ is denoted by $\pi(R)$. Each $i \in I$ learns t, q, p, ω_i, and ψ_i (where ω_i and ψ_i are private information) when he enters the game. For each $t \geq 1$ and each $i \in I$, the history h_i^t consists of the time t and the details of i's interactions up to t (but not i's action in period t, if he takes

[8] In the computer science literature, a reward scheme's robustness to misreporting in this spirit is often referred to as Sybil-proofness, robustness to local false-name manipulations, or robustness to local splits (see, e.g., Emek et al., 2011; Babaioff et al., 2012).

one). Let H_i denote the set of histories following which agent i has to take an action. Agent i's strategy $\sigma_i : H_i \rightarrow \{0,1\}$ maps histories to actions, where following each history $h \in H_i$, agent i either decides whether or not to make an offer (to become a distributor), or decides whether or not to accept one. We shall assume that if an agent is indifferent between making an offer and not doing so (respectively, accepting and rejecting an offer), then he makes (respectively, accepts) it.

Discussion: Modeling assumptions

Meeting process. We borrow the meeting process from the applied statistics literature, where it is referred to as the *random recursive tree model* (for a textbook treatment, see Drmota, 2009). Underlying this process are the assumptions that there is a deterministic date at which the game ends and that the number of entrants in each period t is both deterministic and independent of t. Our main results do not depend on any of these assumptions. Nonetheless, we use this process since it allows us to convey the main messages while keeping the exposition simple. In order to demonstrate the robustness of our results, in Appendix A we modify the process such that conditional on reaching period t, there is a probability $\delta < 1$ that the process continues for an additional period and a probability $1 - \delta$ that the process ends immediately. Our main results hold in this modified *semistationary* setting. The key assumption underlying our results (which is satisfied in both the baseline setting and the modified one) is that there exists a period t such that agents who enter the game after period t are expected to meet a small number of agents.

Word of mouth. The word-of-mouth parameter p adds a realistic aspect to the model. Talkativeness can be interpreted as an agent's tendency to mention the good to others even when he does not have any financial incentive to do so (i.e., when the agent is not himself a distributor). When a talkative agent i (i.e., $\psi_i = 1$) is not a distributor but has heard about the product from some other player j, he will mention it to agents that he encounters and will then refer those agents to player j.

In addition to making the model more realistic, p also provides a natural rationale for recruitment-based compensation in a "legitimate" setting where the SO faces fully rational agents. When $p > 0$, a distributor l who sells a distribution license to an agent $j \in I$ loses his direct access to j's successors but improves the chances that j's successors will purchase the good (e.g., directly from j). Thus, in order to incentivize such information propagation, the SO must use a reward scheme that compensates the distributors for such losses. Surprisingly, as we shall see in the sequel, the talkativeness of non-distributing agents has an additional nontrivial negative effect on the SO's ability to sustain a pyramid scam when he faces boundedly rational agents.

3 Pure Pyramid Scams ($q = 0$)

In order to capture the idea that the only "product" that is being traded in a real-world pyramid scam is the right to recruit others to the pyramid, we set $q = 0$. Thus, it is commonly known that the only products that are being traded in the model are distribution licenses (when an agent accepts an offer to become a distributor, we say that he purchases a distribution license from the distributor who made the offer). Intuitively, such a market should not exist as trade in distribution licenses does not add value. If $q = 0$ and there exists a scheme R such that $\pi(R) > 0$, then we say that the SO is *able to sustain a pyramid scam*. The next result establishes that when all of the agents are fully rational, the SO cannot sustain such a scam.

Proposition 1 *Let $q = 0$. There exists no IC reward scheme R such that the SO makes a strictly positive expected profit in a subgame perfect Nash equilibrium of $\Gamma(R)$.*

Since $q = 0$, reward schemes induce zero-sum transfers between the agents and the SO. Proposition 1 then follows directly from classic no-trade arguments (Tirole, 1982).

Our main objective is to understand the forces and compensation plans that enable pyramid scams to operate. As Proposition 1 shows, it is impossible to do so by means of the classic rational expectations model and we shall therefore depart from this model. We shall weaken the Nash equilibrium assumption that agents have complete understanding of the other agents' behavior in every possible contingency, an assumption that might be too extreme in complicated settings such as the present one.

3.1 The behavioral model

Jehiel (2005) suggests an elegant framework that incorporates partial sophistication into extensive-form games. We adopt this framework and use *analogy-based expectations equilibrium* to solve the model. In an analogy-based expectations equilibrium, different contingencies are bundled into analogy classes. The Nash equilibrium requirement that the agents know the other agents' behavior in every possible contingency is replaced with a milder one: the agents are required to hold correct beliefs about the other agents' *average behavior* in every analogy class. The fact that the agents' beliefs are statistically correct enables us to interpret these beliefs as a result of learning from *coarse feedback* about behavior in similar games that were played in the past. For instance, in the context of the present article, the SO can provide the agents with information about the average "sales force" performance in schemes that he has organized in the past.

In the model, there are two types of strategic decisions: whether or not to purchase a license, and whether or not to sell one. Denote by M_1 (respectively, M_2) the set of histories after which *agents* make the former (respectively, latter) decision. For each $i \in I$, denote by M_1^{-i} and M_2^{-i} the subsets of M_1 and M_2 that exclude histories after which i moves. We use $\sigma = (\sigma_i)_{i \in I \cup \{SO\}}$ to denote a profile of strategies, and $r_\sigma(h)$ to denote the probability of reaching $h \in M_1 \cup M_2$, conditional on σ being played. For each $h \in M_1 \cup M_2$, we use $\sigma(h) = 1$ (respectively, $\sigma(h) = 0$) to denote that the agent who moves following the history h makes an offer or accepts one (respectively, makes no offer or rejects one).

For each $k \in \{1,2\}$ and $i \in I$, let $\beta\left(M_k^{-i}\right)$ denote agent i's *analogy-based expectations* of his opponents' behavior in M_k^{-i}. Let $\beta_i := \left(\beta\left(M_k^{-i}\right)\right)_{k \in \{1,2\}}$ and $\beta := (\beta_i)_{i \in I}$.

Definition 1 *Agent i's analogy-based expectations β_i are said to be consistent with σ if for every $k \in \{1,2\}$, it is $\beta\left(M_k^{-i}\right) = \frac{\sum_{h \in M_k^{-i}} r_\sigma(h)\sigma(h)}{\sum_{h \in M_k^{-i}} r_\sigma(h)}$ whenever $r_\sigma(h) > 0$ for some $h \in M_k^{-i}$.*

In words, the consistency of the agents' analogy-based expectations implies that these expectations *match the other agents' average behavior* under σ. The definition of consistency corresponds to the definition of *weak consistency* in Jehiel (2005). The two notions do not place any restrictions on the agents' beliefs about analogy classes that are not reached with strictly positive probability.[9] We can refrain from placing such restrictions as the only equilibria in which M_1^{-i} and M_2^{-i} are not reached with strictly positive probability are "degenerate" ones (e.g., equilibria in which the SO never makes any offers), which are of secondary interest and do not have any effect on our results.

An important feature of consistency is that histories are weighted according to the likelihood of their being reached. To illustrate this, let $p = 0$, and consider a profile σ such that agents always (respectively, never) purchase a distribution license when they are offered the opportunity to do so in period 1 (respectively, in each period $t > 1$). Further, assume that under σ, in each period $t \leq 3$ (respectively, $t > 3$), every $i \in D_t$ makes (respectively, does not make) an offer when he meets an agent. Note that the third entrant will receive an offer to purchase a license with probability $\frac{2}{3}$ since there is

[9] I thank Erik Eyster for this point.

a probability of $\frac{1}{3}$ that he will meet the second entrant. Consistency implies that each agent i's analogy-based expectations $\beta\left(M_1^{-i}\right)$ are the proportion of offers accepted by the other agents under σ in expectation. Hence,

$$\beta\left(M_1^{-i}\right) = \frac{\frac{n-1}{n}}{\frac{n-1}{n} + \frac{n-1}{n}\left(1 + \frac{2}{3}\right)} = \frac{3}{8} > \frac{1}{n} \tag{1}$$

for each $i \in I$. The fraction $\frac{n-1}{n}$ in (1) results from the fact that i's analogy-based expectations are defined on M_1^{-i}, which does not include the histories in which i plays (recall that with probability $\frac{1}{n}$ he is drawn to be the t-th entrant).

Definition 2 *A strategy σ_i is a best response to β_i if it is a best response to the belief that for each $k \in \{1, 2\}$, in each $h \in M_k^{-i}$, $\sigma(h) = 1$ is played with probability $\beta\left(M_k^{-i}\right)$.*

Definition 3 *The pair (σ, β) forms an analogy-based expectations equilibrium (ABEE) if β_i is consistent with σ and σ_i is a best response to β_i for each $i \in I$.*

Observe that the solution concept does not impose any constraint on the SO's strategy σ_{SO}. The SO chooses a set of periods in which he makes offers to agents whom he interacts with and we allow him to use suboptimal strategies. He can benefit from using a suboptimal strategy as such a strategy will influence β. Thus, allowing the use of suboptimal strategies strengthens our impossibility results. It should be stressed that our possibility results do not make use of suboptimal strategies.

3.2 Existence and structure of pyramid scams

The main result of this section (Theorem 1) shows that the SO cannot sustain a pyramid scam by means of a reward scheme that compensates the distributors only for people whom they directly recruited to the pyramid (i.e., for the number of licenses they sold). That is, underlying a pyramid scam there must be a scheme that compensates the distributors based on at least two levels of recruitments. Theorem 2 provides necessary and sufficient conditions—on the number of potential participants n and their tendency to propagate information by word of mouth p—under which the SO can sustain a pyramid scam. Theorem 3 establishes that the necessary condition of Theorem 1 is tight and Proposition 2 shows that, for every $\alpha > 0$, the results hold in a setting in which a fraction $\alpha > 0$ of the agents are analogy-based reasoners and a fraction $1 - \alpha$ are fully rational.

We start our analysis by asking whether or not it is possible to sustain a pyramid scam by means of a reward scheme that compensates the distributors only for people whom they recruit to the pyramid directly. We shall refer to such a reward scheme as a one-level scheme.[10]

Definition 4 *A reward scheme R is said to be a one-level scheme if $a_\tau^R = 0$ and $b_\tau^R = 0$ for every $\tau > 1$.*

Theorem 1 *Let $q = 0$. There exists no IC one-level scheme R such that $\pi(R) > 0$.*

In a one-level scheme's induced game, purchasing a license is, essentially, betting on the number of distribution licenses that one will sell. The proof of Theorem 1 shows that agents who understand the other agents' average behavior do not overestimate their own ability to sell licenses by much, if at all. Even though the agents wrongly believe that they can recruit other agents to the pyramid in the late stages of the game, their mistakes are not sufficiently big for them to take the bets that the SO, subject to the incentive compatibility constraint, can offer.

The main challenge in the proof is to show that, in any conjectured ABEE, the last entrant who is supposed to purchase a license cannot "analogy-based" expect to sell more than one license (which, if the reward scheme is IC, is a necessary condition for him to purchase a license).

To get a clear intuition of the above, suppose for a moment that $p = 0$ such that the distributors cannot access their successors' successors. Moreover, assume that the agents purchase licenses in an ABEE and that their ABEE play is symmetric. Since the likelihood of meeting the new entrant goes down as time progresses, in this ABEE, there is a period $k^\star$ such that each agent who receives an offer to purchase a license in each $t \leq k^\star$ accepts it and each agent who receives such an offer after $k^\star$ rejects it. Nonetheless, the distributors continue making offers late in the game (i.e., after $k^\star$) as they wrongly believe that the other agents might accept them. Denote the expected number of offers that each distributor makes after period $k^\star$ by v and note that all of these offers will be rejected. Thus, each offer that is accepted at $t \leq k^\star$ results in a distributor who, in expectation, makes v offers that are rejected after period $k^\star$. It follows that each agent i's analogy-based expectations $\beta\left(M_1^{-i}\right)$, which, roughly speaking, are the proportion of accepted offers, cannot exceed $\frac{1}{1+v}$ and that the $k^\star$-th entrant cannot analogy-based expect to sell more than $\frac{v}{1+v}$ licenses.

When $p > 0$, there is an additional effect that is worth mentioning: the $k^\star$-th entrant underestimates the number of offers that he will make (i.e., he believes that he will make $\tilde{v} \leq v$ offers). To grasp this effect, suppose that the $k^\star$-th entrant, agent

[10]The sales commissions do not play a role in this section. Nonetheless, for completeness, we include them in the definition of one-level schemes.

i, makes an offer to an agent j in period $t > k^\star$. This offer will be rejected by j as it is made too late in the game. If j is talkative, then, after he rejects the offer, he will refer his successors to i who will be able to make them some additional offers. Agent i underestimates the likelihood that j will reject his offer and refer other agents to him as he thinks that j accepts offers with probability $\beta\left(M_1^{-i}\right) > 0$ regardless of their timing. Thus, when $p > 0$ (i.e., when agents may be talkative), the $k^\star$-th entrant "analogy-based expects" to sell fewer than $\frac{v}{1+v}$ licenses.

We wish to stress that the result presented above does not depend on the subtleties of the network formation process (e.g., the finite number of periods or the assumption that the number of entrants in each period is independent of t). Rather, the result relies on the fact that there is a period $k^\star$ from which point onward the agents do not purchase licenses. As long as the number of entrants in each period is non-increasing as time progresses from some point in time, a period from which point onward the agents reject every offer will exist in every ABEE and the result of Theorem 1 will hold.

Note also that, a priori, the agents' ABEE behavior need not be symmetric. In an asymmetric ABEE, the agents' analogy-based expectations can differ, as the proportion of offers accepted by the members of $I - \{i\}$ can be different from the proportion of offers accepted by the members of $I - \{j\}$. Moreover, unlike in the symmetric case, some of the agents' analogy-based expectations can be strictly greater than the proportion of offers that are accepted by the members of I. For these reasons, the analysis of the asymmetric case is more challenging and requires a few additional steps, which are elaborated in the proof.

For each $i \in I$, we shall refer to the distributors in $G_i - \{i\}$ as i's *downline* and say that distributor j is a member of the λ-th level of i's downline if j is a member of i's downline and there are $\lambda - 1$ distributors on the path connecting i to j. So far, we have shown that the SO cannot sustain a pyramid scam by means of a one-level scheme. We shall now provide necessary and sufficient conditions under which he can sustain such a scam by means of a reward scheme that compensates the distributors for their recruits and the recruits of other members of their downlines.

Theorem 2 *Fix $q = 0$. There exists a number $n^\star$ such that for every $n > n^\star$ there exists a number $p^\star(n) < 1$ such that:*

1. *If $p \leq p^\star(n)$, then there exists an IC reward scheme R such that $\pi(R) > 0$.*

2. *If $p > p^\star(n)$, then there exists no IC reward scheme R such that $\pi(R) > 0$.*

Theorem 2 establishes that the SO can sustain a pyramid scam if both n is sufficiently large and the talkativeness parameter p is sufficiently small. The talkativeness parameter p can be interpreted as a property of the good that disguises the underlying scam. A high value of p corresponds to contagious (or unique) goods in the sense that people are excited to talk about them with their friends and acquaintances. Theorem 2 shows that an SO who wishes to initiate a pyramid scam will prefer to disguise it by means of a good that does not provoke such word-of-mouth advertisement.

Before we discuss the role that p plays in this result, it is perhaps useful to provide some intuition of the existence result. This intuition resembles the one behind the ABEE analysis of the finite-horizon centipede game (Jehiel, 2005), which we shall discuss in the concluding section. The present setting is not stationary and this leads to nonstationary equilibrium behavior, as the agents accept offers until some period $k^\star$ (if at all) and reject offers that arrive later in the game. A "rational" economic agent will not join the pyramid in period $k^\star$, knowing that in the later stages of the game it is no longer beneficial to purchase a license. However, the analogy-based reasoners in our model view the other agents' behavior as if it were time-invariant: each agent i wrongly believes that the others always accept offers with the average probability $\beta\left(M_1^{-i}\right)$, even when it is no longer beneficial to join the pyramid. This overoptimistic belief (i.e., relying on the other agents' average behavior rather than on their marginal behavior) is what makes some of the agents join the pyramid in the hope of benefiting at the expense of the SO and future entrants.

The fact that the talkativeness parameter has a negative effect on the SO's ability to sustain a pyramid scam may not be so intuitive at first glance. Observe that the larger p is, the better the distributors' access to their successors' successors, which allows them to enjoy a larger potential clientele. This may suggest that p has a positive effect on the agents' willingness to participate in a pyramid scam. However, p has an additional negative effect on the agents' "feedback" (i.e., the analogy-based expectations), which may not be as transparent as the above-mentioned positive effect.

The intuition for the negative effect is as follows. The agents' feedback is, roughly speaking, the proportion of accepted offers and, therefore, it is determined by the ratio of accepted to rejected offers. In equilibrium, the agents accept offers to join the pyramid early in the game and reject such offers in the game's later stages. This implies that the ratio of non-distributors to distributors is higher in the later stages of the game such that, compared to the early entrants, agents who enter late in the game are far more likely to encounter a non-distributor. The higher p is, the more likely each non-distributor is to be talkative and refer the late entrants whom he meets to a distributor who will make them an offer (which they will reject, as it is no longer beneficial to join the pyramid). In conclusion, a higher p implies that a larger number of offers are made late in the game and, since these offers are rejected, a higher value of p worsens the agents' analogy-based expectations to the extent that, when p is large, the SO cannot sustain a pyramid scam.

In light of the existence result of Theorem 2, it is natural to ask what is the minimal number of levels of the distributors' downlines that the compensation must be made contingent on for a pyramid scam to be sustained. Theorem 3 establishes that the SO might be able to sustain a pyramid scam by means of a two-level reward scheme (i.e., a scheme R such that $a_{\tau}^{R} = 0$ and $b_{\tau}^{R} = 0$ for every $\tau > 2$).

Theorem 3 *Fix $q = 0$. There exists a number $n^{\star\star} \geq n^{\star}$ such that for every $n > n^{\star\star}$ there exists a number $p^{\star\star}(n) < 1$ such that if $p \leq p^{\star\star}(n)$, then there exists an IC two-level scheme R such that $\pi(R) > 0$.*

What is the difference between a one-level scheme and a multilevel scheme? In a one-level scheme's induced game, the agents bet on their own ability to sell licenses. The agents' misspecified model of the other agents' behavior leads them to mispredict this variable. However, as we showed in Theorem 1, these prediction errors are "too small" for the SO to overcome the incentive-compatibility constraint and sustain a pyramid scam. Multilevel schemes induce more complicated bets that require the agents to predict multiple variables (e.g., in a two-level scheme's induced game the agents have to estimate not only their own ability to recruit others to the pyramid but also their recruits' ability to do so). The agents mispredict all of these variables. While each of these prediction errors is small, their accumulation allows the SO to overcome the incentive constraint and sustain a pyramid scam.

3.3 A mixture of fully and boundedly rational agents

So far, we have assumed that either all of the agents are fully rational or the whole population of agents consists of analogy-based reasoners. Let us examine a population that consists of a mixture of the two types. We assume that $\lceil \alpha n \rceil$ of the agents are analogy-based reasoners and that $n - \lceil \alpha n \rceil$ of the agents are fully rational. The fully rational agents have accurate beliefs about the other players' behavior in every contingency. Thus, we expect them to reject offers to join a pyramid even in instances where analogy-based reasoners are happy to accept them. Clearly, the impossibility results of Theorems 1 and 2.2 hold for every $\alpha > 0$. Showing that the positive results of Theorems 2.1 and 3 hold for every $\alpha > 0$ is a more challenging task.

Before proceeding to the result, it is necessary to slightly modify the solution concept in order to incorporate the rational agents' behavior. We shall assume that the analogy-based reasoners best respond to analogy-based expectations that are consistent with the equilibrium strategies of all of the agents. The rational agents best respond to the equilibrium play using the accurate (i.e., not coarse) model of the world.

Proposition 2 *Fix $q = 0$ and an arbitrary $\alpha \in (0, 1]$. There exists a number $n(\alpha)$ such that for every $n > n(\alpha)$, there exists a number $\hat{p}(n) < 1$ such that if $p \leq \hat{p}(n)$, then there exists an IC two-level scheme R such that $\pi(R) > 0$.*

Proposition 2 establishes that the SO can sustain a pyramid scam even in the presence of fully rational agents. The intuition is that by taking n to be large it is always possible to "compensate" for the existence of the fully rational agents who reject offers in instances in which the analogy-based reasoners accept them.

When $\alpha < 1$ is sufficiently close to 1, there are cases in which rational agents purchase licenses in equilibrium. In these cases they correctly expect to recruit analogy-based reasoners later in the game. It should be noted that fully rational agents cannot be "scammed"; when they participate in a pyramid scam, they make a positive expected payoff at the expense of the analogy-based reasoners who, on average, incur losses.

It is worth noting that the presence of *fully rational agents can protect the analogy-based reasoners* from deceptive practices. The fact that they reject offers in instances in which the analogy-based reasoners will accept them worsens the analogy-based reasoners' expectations to the extent that, for small values of α (fixing an arbitrary population size $\bar{n}$), the SO cannot sustain a pyramid scam regardless of p.

3.4 Example: A multilevel scheme maximizes the SO's profit

We now examine the schemes that maximize the SO's expected ABEE profit and illustrate that these schemes may compensate the distributors based on strictly more than two levels of their downlines. Moreover, this example demonstrates that agents who join the pyramid early can make a strictly positive expected payoff at the expense of the agents who purchase their license later in the game.

We shall say that an IC scheme R is *profit-maximizing* if there exists no IC scheme R' such that $\pi(R') > \pi(R)$. Observe that due to their risk neutrality, both the agents and the SO can benefit from raising the stakes of the bets between them (i.e., multiplying the recruitment commissions and the entry fees by a constant $\gamma > 1$ does not change the set of ABEEs but increases the SO's expected profit). In order to bound these stakes such that a profit-maximizing scheme will exist, we shall modify the model by assuming that the maximal amount that each agent can pay for a license is $B > 0$. Under this assumption, if the SO can sustain a pyramid scam, then every profit-maximizing scheme charges a fee of B. Without loss of generality, we shall restrict our attention to such schemes.

Let $p = 0$ such that each distributor finds it optimal to make an offer to each agent whom he meets. It can be shown that, when $p = 0$, in every ABEE in which the agents purchase licenses, there is a period $k^\star$ such that each agent who receives an offer in each $t \leq k^\star$ accepts it, and each agent who receives an offer after $k^\star$ rejects it. Each distributor meets, in expectation, $\sum_{j=k^\star+1}^{n} \frac{1}{j}$ agents after period $k^\star$. Since $p = 0$, distributors do not interact with agents whom they are not directly linked to. Thus, each accepted offer results in a distributor who makes, in expectation, $\sum_{j=k^\star+1}^{n} \frac{1}{j}$ offers after period $k^\star$. Recall that offers are not rejected prior to $k^\star$. At the optimum, the SO does not make offers after period $k^\star$ as such offers are rejected and negatively affect the agents' analogy-based expectations. It follows that the proportion of accepted offers is $\frac{1}{1+\sum_{j=k^\star+1}^{n} \frac{1}{j}}$. The symmetry of the agents' strategies implies that $\beta\left(M_1^{-i}\right) = \frac{1}{1+\sum_{j=k^\star+1}^{n} \frac{1}{j}}$ for each $i \in I$.

Observe that $k^\star$ pins down the agents' analogy-based expectations (which determine their willingness to pay for a license) and the SO's revenue (which is the number of distributors multiplied by B). The profiles of strategies and analogy-based expectations described above form an ABEE of multiple IC schemes' induced games. While the SO's revenue does not depend on which of these reward schemes is used, his expected cost does depend on which scheme is used. The scheme that minimizes the SO's costs given such an ABEE has to compensate the $k^\star$-th entrant (i.e., make him "analogy-based expect" rewards of B) in the least costly way.

For a fixed profile of strategies σ, the "dual" problem is essentially how to minimize a linear function $\kappa_1 a_1^R + \kappa_2 a_2^R + ... + \kappa_{n-k^\star} a_{n-k^\star}^R$, subject to the incentive-compatibility constraints that $a_1^R, ..., a_{n-k^\star}^R \leq B$, subject to $a_1^R, ..., a_{n-k^\star}^R \geq 0$, and subject to the $k^\star$-th entrant being willing to pay B for a license. Each weight κ_τ represents the expected cost that is associated with an increase in the commission a_τ^R given σ. The $k^\star$-th entrant's willingness to pay for a license is $\sum_{\tau=1}^{n-k^\star} \beta\left(M_1^{-i_{k^\star}}\right)^\tau l_\tau a_\tau^R$, where l_τ is the expected number of agents in the subtree of G rooted at the $k^\star$-th entrant.

In order to find an IC profit-maximizing scheme, it is sufficient to find a cost-minimizing scheme for every possible value of $k^\star \in \{1, ..., n\}$ and compare the SO's expected profit in the corresponding ABEEs.[11] For $n = 100$, $B = 1$, and $c = 0$, the SO's expected profit is maximized by means of a scheme R that pays $a_1^R = 0.823$ for

[11] For each value of $k^\star$, it is also necessary to consider all of the SO's feasible strategies, as he may decide not to make offers in some of the first $k^\star$ periods of the game. The SO's choices in the first $k^\star$ periods affect the costs associated with each commission (i.e., κ_τ) but not the agents' analogy-based expectations.

direct recruitments and $a_\tau^R = 1$ for every $\tau > 1$. In the ABEE of $\Gamma(R)$ in which the SO's expected profit is maximized, the first four entrants purchase a license (i.e., $k^\star = 4$) and the SO's expected profit is 1.922. The first entrant makes an expected payoff of 0.055 at the expense of the other participants, who incur losses. If the SO were restricted to using a two-level scheme instead of a multilevel one, then the maximal expected profit he could obtain in an ABEE would be 1.864. Hence, in this case, the SO is strictly better off using a multilevel reward scheme rather than a two-level one.

4 Multilevel Marketing of Genuine Goods ($q > 0$)

In the previous section, the only "products" that were being traded were distribution licenses. The SO's profit was a result of the agents' partial sophistication. In this section, we shall explore a "legitimate" world in which the good is intrinsically valued such that, in addition to benefiting from selling licenses, the SO benefits from selling the good.

We shall assume that $q > 0$ and analyze the schemes that maximize the SO's expected equilibrium profit in two settings that correspond to the ones that we have studied in Section 3. In Section 4.1, the agents are fully rational and do not err, such that the SO cannot benefit from deceptive practices. In Section 4.2, the agents are boundedly rational and the SO can benefit from their mistakes as well as from their sales. In both settings, we shall focus on two controversial components of the reward schemes, namely, entry fees and recruitment commissions. We shall investigate whether or not these components must be used in order to maximize the SO's expected profit.

4.1 MLM with fully rational agents

We set $q > 0$ in order to capture that the good is genuine and we use subgame-perfect Nash equilibrium (SPE) in order to solve the model, as the agents are fully rational. As in the example in the previous section, we shall say that an IC reward scheme R is profit-maximizing if there exists no IC reward scheme R' such that $\pi(R') > \pi(R)$.

Theorem 4 shows that if the SO produces a genuine good and faces fully rational agents, then profit-maximizing schemes need not charge entry fees or pay recruitment commissions.

Theorem 4 *Let $c > 0$. There exists an IC profit-maximizing scheme $R^\star$ such that $\phi^{R^\star} = 0$ and $a_\tau^{R^\star} = 0$ for every $\tau \geq 1$.*

The proof shows that for each scheme R, there exists an IC scheme $R^\star$ that does not pay recruitment commissions, does not charge entry fees, and where $\pi(R^\star) \geq \pi(R)$. The scheme $R^\star$ pays each distributor j, in expectation, $p^{\tau-1} b_1^{R^\star}$ for every unit purchased by an agent i such that $d_G(j,i) = \tau$, regardless of the number of distributors on the path connecting j and i. Thus, $R^\star$ leaves each distributor indifferent between recruiting and selling the good directly. This indifference has two implications. First, in expectation, each distributor is paid as if he does not recruit at all. Each distributor can secure this payment in $\Gamma(R)$ by selling the good without recruiting. Hence, the SO's expected transfer to each distributor is less under $R^\star$ than it is under R. The second implication of the distributors' indifference is that any profile of strategies that constitutes an SPE of $\Gamma(R)$ also constitutes an SPE of $\Gamma(R^\star)$. In conclusion, the scheme $R^\star$ sustains the SO's preferred SPE behavior in $\Gamma(R)$ while maintaining a lower overhead.

It is worth mentioning that except in a few extreme cases, an IC scheme R that charges entry fees or pays recruitment commissions cannot be profit maximizing. In fact, if the SPE of $\Gamma(R)$ that induces $\pi(R)$ results in two or more distributors, then the distributors' expected compensation is strictly less under $R^\star$ than it is under R such that $\pi(R^\star) > \pi(R)$. Thus, in every case where multilevel marketing is relevant (i.e., the SO uses at least two distributors), schemes that charge entry fees or pay recruitment commissions are not profit maximizing.

Interestingly, the compensation under $R^\star$ is contingent on *more than two levels of retail sales*. The commission $b_1^{R^\star}$ is meant to cover the cost c. The purpose of $b_2^{R^\star}$ is to incentivize information propagation. This incentive is necessary since a distributor who recruits an agent is basically recruiting a competitor, as he loses access to that agent's successors. However, $b_2^{R^\star}$ may not suffice to incentivize recruitment of new competitors since it does not compensate the distributors for the fact that these competitors may recruit additional competitors themselves. For example, a distributor d who recruits an agent i loses access to i's successors. As long as i does not recruit anyone, $b_2^{R^\star}$ compensates d for these losses. However, if i recruits an additional agent j, then the recruitment reduces i's sales as he can no longer sell to j's successors and this may negatively affect d's reward, which, in a two-level scheme, is based on i's sales but not on j's sales. The higher-order commissions (e.g., $b_3^{R^\star}$) compensate the distributors for the fact that their recruits may recruit new recruits (who may in turn recruit, and so on).

So far in this section, we have shown that recruitment commissions and entry fees are inconsistent with profit maximization when the good is intrinsically valued and the agents are fully rational. In the previous section, we established that pure pyramid scams are based on these two components. There are two differences between the settings of the two sections. First, in Section 3.2 the agents are boundedly rational while in Section 4.1 they are fully rational and, therefore, cannot fall victim to deceptive practices. The second difference is that in Section 4.1 the SO can profit from selling the good while in Section 3.2 it is impossible to do so. In the next section we shall complete the analysis by investigating a setting in which the SO has two potential sources of income: selling the good and scamming the agents.

4.2 MLM with genuine goods and analogy-based reasoners

We now study a setting in which the agents are boundedly rational such that the SO can exploit them. Unlike in Section 3.2, we shall assume that the agents are willing to pay for the good (i.e., $q > 0$). This creates an additional source of income for the SO. As in the previous section, we shall examine whether or not profit-maximizing schemes pay recruitment commissions or charge entry fees.

We shall simplify the analysis by assuming that $p = 0$ such that the distributors do not have access to their successors' successors. Moreover, we shall restrict attention to symmetric ABEEs in this section.[12] An ABEE (σ, β) is said to be *symmetric* if there is a set of periods $\mathcal{T}$ such that, under σ, in every $t \in \mathcal{T}$, each agent $i \in I$ accepts every offer to purchase a license and in every $t \notin \mathcal{T}$, each agent $i \in I$ rejects every offer to purchase a license. Let us modify $\pi(R)$ to be the highest expected profit that the SO can obtain in a symmetric ABEE of $\Gamma(R)$ and change the definition of a profit-maximizing scheme accordingly: an IC reward scheme R is profit maximizing if there exists no IC reward scheme R' such that $\pi(R') > \pi(R)$.

In order to guarantee that profit-maximizing schemes exist, as in the example in Section 3.4, we shall assume that each agent cannot pay more than $B > 0$ for a license. As we shall see in Proposition 3, unlike in the example, profit-maximizing schemes do not necessarily charge a fee of B.

Proposition 3 shows that even though the agents are vulnerable to deceptive practices, when the potential gains from sales are large, schemes that pay recruitment commissions or charge entry fees are not profit maximizing.

Proposition 3 *Fix $q > 0$, $c > 0$, and $p = 0$. There exists a number $\hat{n}$ such that if $n > \hat{n}$ and R is a profit-maximizing IC scheme, then $\phi^R = 0$ and $a^R_\tau = 0$ for every $\tau \geq 1$.*

[12] It is possible to show that this restriction entails no loss of generality when $p = 0$.

When n is large, the potential gains from trade are large as well, which makes the distributors more valuable to the SO. The first part of the proof confirms this intuition. It is shown that for any fixed $\tau \in \mathbb{N}$, if n is sufficiently large and R is profit maximizing, then the number of agents who purchase a license in every ABEE of $\Gamma(R)$ that induces $\pi(R)$ is greater than τ with a probability arbitrarily close to 1.

In the second part of the proof we study the SO's dual problem. We fix an arbitrary fee ϕ and a profile of strategies σ, and examine the schemes that minimize the SO's expected cost out of the set of IC schemes that charge ϕ and where σ is part of an ABEE of their induced game. Denote a scheme that solves this problem for ϕ and σ by $R(\phi, \sigma)$. We show that for every $\phi > 0$ and every profile σ in which the number of agents who purchase a license is sufficiently large with a sufficiently high probability, $\pi(R(0, \sigma)) > \pi(R(\phi, \sigma))$. This means that charging entry fees is inconsistent with profit maximization when many agents purchase a license. Together with the finding of the first part of the proof, it implies that for large values of n, schemes that charge entry fees are not profit maximizing. The incentive-compatibility property implies that IC schemes that pay recruitment commissions are not profit maximizing either.

To obtain intuition for why schemes that charge entry fees are suboptimal when n is large, let us consider the effects of charging a fee on the SO's expected profit. First, it raises the SO's revenue per distributor. Second, the positive fee relaxes the incentive-compatibility constraint and allows the SO some flexibility since it allows him to compensate the distributors by means of recruitment commissions that compared to the sales commissions induce smaller costs for the SO. Third, the fee raises the cost of becoming a distributor $\phi + c$ such that maintaining the agents' willingness to purchase a license requires raising the commissions and the SO's costs.

When the number of distributors is large, the third effect dominates the first two effects. The reason for this is that the bulk of the sales and recruitments are done by distributors whose distance from the SO is large. The SO has to pay multiple commissions (e.g., $a_1 + a_2 + a_3 + ...$) for each such recruitment or sale such that raising the commissions to compensate the distributors for the higher fee becomes very costly when the number of distributors becomes large. Hence, if the number of distributors is likely to be large, the SO's expected profit is maximized by means of a reward scheme that does not charge entry fees.

Let us change the order of quantifiers and examine the limit at which the SO faces a small demand. Fix n and consider a case in which q is close to 0. The maximal expected profit that the SO can make without charging entry fees is qn. Thus, if $n > n^\star$ and $p \leq p^\star(n)$, then profit-maximizing reward schemes charge entry fees, pay recruitment commissions, and, in expectation, the participants in their induced games incur losses.

5 Comparison to Non-common Priors Models

A natural modeling assumption in the context of bubbles and pyramid schemes is that people underestimate the finiteness of the process. We now examine a case in which different agents hold different prior beliefs about the length of the game. Let us relax the assumption that the number of periods n is fixed and assume instead that each agent i believes that, for each period $t \in \mathbb{N}$, conditional on the game reaching period t, the game continues for an additional period (and a new agent enters) with a probability of δ_i, and that it terminates at t with a probability of $1 - \delta_i$. Assume that it is commonly known that the agents' discount factors are drawn from a distribution with a support $[\underline{\delta}, \bar{\delta}]$, where $\bar{\delta} < 1$. A higher discount factor represents an agent who is more optimistic about the number of future entrants. The next result establishes that if there are no additional departures from the classic rational expectations model, then the SO cannot sustain a pyramid scam.

Proposition 4 *Let $q = 0$. There exists no IC reward scheme R such that $\pi(R) > 0$.*

What is the difference between ABEE and non-common priors? Agents who hold non-common prior beliefs differ in how they evaluate the future, but their evaluation is history-independent. However, in an ABEE, the players' evaluation of the future is affected by the overall behavior of the other players and, in particular, by the events that take place early in the game. This history-dependence prevents the use of standard backward induction arguments and enables pyramid scams to work.

6 Concluding Remarks

Legitimate multilevel marketing and fraudulent pyramid scams are two widespread phenomena. Experts and potential participants often find it hard to distinguish between them. We developed a model that allowed us to draw the boundary between the two based on observable properties of their underlying compensation structures. We provided necessary and sufficient conditions on the potential participants' tendency to spread information by word of mouth, such that it is possible to sustain a pyramid scam even though the potential participants' beliefs are statistically correct. The main result of the paper makes a connection between dubious "passive income" promises and pyramid scams by showing that every pyramid scam has a reward scheme that compensates the participants for at least two levels of recruitments. That is, reward schemes that compensate the participants only for people whom they recruit to the sales force cannot sustain a pyramid scam, but multilevel schemes (and, in particular, two-level schemes) can sustain such a scam.

Our analysis in Section 3.3 showed that even in the presence of fully rational agents, the SO may sustain a pyramid scam. In fact, fully rational agents may join a pyramid scam in its early stages and benefit at the expense of the SO and the later entrants, who, on average, incur losses. However, the presence of fully rational agents makes it more difficult for the SO to sustain a pyramid scam. Fully rational agents reject offers to join a pyramid in instances in which boundedly rational agents would be happy to join and, therefore, the presence of fully rational agents worsens the boundedly rational agents' equilibrium feedback (i.e., their analogy-based expectations) to the extent that it may be impossible for the SO to sustain the pyramid scam. Thus, the presence of fully rational agents may protect the boundedly rational agents from participating in a pyramid scam.

Pyramid scams are based on two components, namely, recruitment commissions and entry fees. We established that legitimate companies (i.e., ones that promote genuine goods) that face fully rational consumers cannot benefit from basing their compensation on these pyramidal components. In such instances, it is possible to maximize the company's expected profit while compensating the sales force based only on sales. In fact, even when such a company faces boundedly rational agents who are vulnerable to deceptive practices, it may find it suboptimal to use these pyramidal components if the demand for its product is sufficiently large.

The problem of incentivizing the recruitment of potential competitors that emerges in the model has received considerable attention outside the economics literature. Cebrian et al. (2011) examine this problem in the context of the DARPA Network Challenge, which was announced by the Defense Advanced Research Projects Agency in 2009. Babaioff et al. (2012) study such a problem in the context of Bitcoin, where network nodes obtain a reward if they are the first to certify a transaction. From the designer's perspective, it is best if as many nodes as possible attempt to answer the query; however, the nodes may have insufficient incentives to spread the information and solicit other nodes to compete for the reward. In these settings, multilevel schemes can mitigate the problem and incentivize recruitment of potential competitors.

We shall conclude by discussing a few extensions and modifications of the model.

Uncertain arrival time

Abreu and Brunnermeier (2003) study a model in which a finite process creates a bubble that bursts after a synchronized attack by a sufficient number of investors or at the end of the process. The investors in their model become aware of the bubble sequentially, and face *uncertainty about the time at which the bubble started* and how many other investors are aware of the bubble. They show that the bubble may persist long after all of the investors are aware of its existence.

As in the case of a bubble, time plays a major role in pyramid scams since, potentially, agents who join early can benefit at the expense of those who join later. However, uncertainty about arrival time *cannot lead to participation* in a pyramid scam when there are no deviations from the classic rational expectations model. The reason that such uncertainty sustains a bubble in Abreu and Brunnermeier's model is that their trading game is not a zero-sum game. Underlying the bubble in their model is an exogenous process that represents behavioral traders. Rational investors who ride the bubble profit at the expense of those behavioral traders, unlike pyramid-scam participants who, on average, incur losses.

What would be the implications of assuming that agents are uncertain about the time at which they enter the game, in addition to assuming that their reasoning is coarse? Adding a small noise to the model would not change the essence of our main results. However, if the agents' information about their time of entry were to become very coarse (e.g., the extreme case in which they assign equal probability to each $t \in \{1, ..., n\}$), then it would be impossible for the SO to sustain a pyramid scam. The reason for this, loosely speaking, is that when an agent does not know his position in the game tree, best responding to the other agents' average behavior (as in an ABEE) is a relatively small mistake.

Relation to the centipede game

It is well known that in the unique SPE of the finite-horizon centipede game, players always *stop* even though they can benefit if they *continue* for a few rounds. Jehiel (2005) resolves this paradox by showing that it is possible to sustain an ABEE in which players continue. If the centipede game is sufficiently long, then under the coarsest partition there exists an ABEE in which the players continue until the last round. There are several differences between the games induced by the different reward schemes in our model and the centipede game that not only allow us to answer questions related to the application, but may also lead to results that are fundamentally different, as is illustrated in our impossibility results in Section 3.

The restrictions on the payoff function that result from incentive compatibility and the requirement that the SO's expected profit be positive, and the effect of p on the agents' analogy-based expectations, may prevent the SO from sustaining a pyramid scam even if the game is arbitrarily long (e.g., if $p = 1$). More importantly, they impose an interesting structure on the participants' compensation when a scam is sustainable.

A more traditional branching process

The network formation model that we used throughout the article assumes that, in each period, the agent who enters the game can meet any of the agents who entered the game previously. A more traditional approach is to assume that new nodes can link only to "leaves" in the existing tree (i.e., agents who enter the game at time $t+1$ can meet only the time-t entrants). Let us study such a process. Assume that in each period $t \in \{1, ..., n\}$, each of the players who entered the game in period $t-1$ meets μ_t new agents. As in our baseline model of Section 2, assume that, for each $t \geq 1$, the period-t entrants are drawn by nature, uniformly at random, from a finite set I.

Let us examine in this setting Theorem 1, which is perhaps the most surprising result of the paper. To keep the exposition simple, let $p = 0$ such that each distributor finds it optimal to make an offer to every agent whom he meets. Let us focus on symmetric profiles of strategies. Consider a profile σ in which every agent who receives an offer at time $t \leq k < n$ accepts it and every agent who receives an offer at $t = k+1$ rejects it. Since $p = 0$, no offer is made after $t = k+1$. Note that every symmetric ABEE in which the SO's expected profit is strictly positive must have this structure. Moreover, as in our baseline model of Section 2, in a symmetric ABEE each agent i's analogy-based expectations $\beta\left(M_1^{-i}\right)$ are equal to the expected proportion of accepted offers.

Under σ, $\mu_1 + \mu_1\mu_2 + ... + \prod_{j=1}^{k+1}\mu_j$ offers are made and $\mu_1 + \mu_1\mu_2 + ... + \prod_{j=1}^{k}\mu_j$ of them are accepted. Hence, for each agent $i \in I$, it is $\beta\left(M_1^{-i}\right) = \frac{\mu_1+\mu_1\mu_2+...+\prod_{j=1}^{k}\mu_j}{\mu_1+\mu_1\mu_2+...+\prod_{j=1}^{k+1}\mu_j}$. Let $\mu_z = min\left\{\mu_2, ..., \mu_{k+1}\right\}$ and consider an agent i who purchases a license in period $z-1$. Agent i believes that, in expectation, he will sell $\mu_z\beta\left(M_1^{-i}\right) < 1$ licenses. If R is an IC one-level scheme, then agent i cannot recover the cost of purchasing a license, which is in contradiction to σ being part of an ABEE.

Multiple schemes

The SO in our model offers one scheme by which the agents can become distributors. In practice, MLM companies sometimes offer several such schemes. Hence, it makes sense to examine whether or not the SO can benefit from offering multiple schemes and having the agents self-select into them. Potentially, the SO might prefer the agents to choose different schemes in different periods as it would allow him to code the nonstationarity of the setting into the rewards.

Whether or not the SO can benefit from offering multiple schemes depends on the agents' level of sophistication. If the SO faces a boundedly rational agent, they might disagree on which scheme is more profitable for the agent. Thus, there are cases where, in each period, a boundedly rational agent mistakenly chooses the scheme that is most profitable for the SO and least profitable for the agent, such that offering multiple schemes renders the SO better off. However, when the SO faces fully rational agents, offering multiple schemes can only render the SO worse off. It is possible to show (the proof is similar to the proof of Theorem 4) that for every set of schemes $\mathcal{S}$, there exists an IC scheme $R^{\star}$ that does not pay recruitment commissions, does not charge entry fees, and where $\pi\left(R^{\star}\right)$ is weakly greater than the SO's expected profit in every SPE of the game that is induced by $\mathcal{S}$.

In conclusion, when the SO faces fully rational agents he cannot increase his profit by means of offering multiple schemes and having the agents self-select into these schemes; however, the SO can benefit from offering multiple schemes for joining the sales force when he faces boundedly rational agents who potentially self-select into the "wrong" schemes.

Appendix A: A semistationary model

Our main objective is to show that the paper's main insights do not depend on the finiteness of the game. We shall focus on Theorems 1–4 and show that similar results hold when there is uncertainty about the length of the game.

Let us relax the assumption that the game has a fixed number of n periods and assume instead that, for each $t \in \mathbb{N}$, conditional on the game reaching period $t \in \mathbb{N}$, there is a probability of $\delta < 1$ that the game continues and a probability of $1-\delta$ that it terminates in period t. Note that we can no longer assume that the set of agents is finite. We shall assume that the set of potential entrants is $I=[0,1]$ and that, as in the main text, in each period $t \in \mathbb{N}$, nature draws one agent $i \in I$ to enter the game, uniformly at random. In order to facilitate the exposition, we shall assume that each agent i's strategy $\sigma_i : \mathbb{N} \rightarrow \{0,1\} \times \{0,1\}$ is a mapping from time to two decisions: whether or not to purchase a license and whether or not to make an offer.

For each $t \in \mathbb{N}$, the average probability that agents accept an offer at t is $\bar{\sigma}_t := \int_{j \in I} \sigma_j\left(t\right) dj$, where $\sigma_j\left(t\right)=0$ (respectively, $\sigma_j\left(t\right)=1$) if j rejects (respectively, accepts) offers he receives in period t. Let $r_{\sigma}(t)$ be the probability that the t-th entrant receives an offer to purchase a license given the profile σ. For each $i \in I$, we shall say that $\beta\left(M_1^{-i}\right)$ is consistent with σ if $\beta\left(M_1^{-i}\right) = \sum_{t=1}^{\infty} r_{\sigma}(t)\,\bar{\sigma}_t$ whenever $r_{\sigma}(t) > 0$ for some[13] $t \in \mathbb{N}$. The consistency of $\beta\left(M_2^{-i}\right)$ is defined in an analogous manner. As in the main text, an ABEE is a pair of profiles (σ, β) such that the agents' analogy-based expectations are consistent with σ and each agent's strategy is optimal w.r.t. his analogy-based expectations. The rest of the modeling assumptions remain as in the main text.

A.1 Fully rational agents

The next result shows that Theorem 4 does not rely on the finiteness of the game. Its proof is identical to that of Theorem 4 except for the following difference: since the number of periods is not finite, we need to show that there is a period $t^{\star} \in \mathbb{N}$ such that in every game that is induced by an IC scheme, from period $t^{\star}$ onward, rejecting every offer to purchase a license is the unique best response of each agent $i \in I$ (regardless of his beliefs about the other agents' behavior). This technical result will allow us to treat the game as one with a finite number of periods.

Proposition 5 *Let $c > 0$ and $q > 0$. There exists an IC profit-maximizing scheme $R^{\star}$ such that $\phi^{R^{\star}} = 0$ and $a_{\tau}^{R^{\star}} = 0$ for every $\tau \geq 1$.*

A.2 Analogy-based reasoners

We now focus on the results of Theorems 1, 2.1, and 3, and show that similar results hold in this semistationary setting.

Proposition 6 *Let $q = 0$. There exists no IC one-level scheme R such that $\pi(R) > 0$.*

As in Theorem 1, the agents do not overestimate their own ability to recruit other agents to the pyramid by much, if at all. The last agent to purchase a license in any conjectured ABEE cannot believe that, in expectation, he will sell more than one license. The only difference between the proofs of Theorem 1 and Proposition 6 is that, in the latter, we need to take into account the probability that the game ends before the agents start rejecting offers. This possibility makes the proportion of accepted offers (and the agents' analogy-based expectations) slightly higher in comparison to the setting in the main text. However, this change does not affect the result.

Proposition 7 shows that the SO can sustain a pyramid scam by means of a two-level reward scheme if p is sufficiently low and δ is sufficiently close to 1.

[13] Since $\int_{j \in I} \sigma_j(t)\,dj = \int_{j \in I - \{i\}} \sigma_j(t)\,dj$, it does not matter whether or not i's analogy-based expectations take his own actions into account.

Proposition 7 *Fix $q = 0$. There exists a number $\delta^\star < 1$ such that for every $\delta > \delta^\star$ there exists a number $p^\star(\delta)$ such that if $p \leq p^\star(\delta)$, then there exists an IC two-level scheme R such that $\pi(R) > 0$.*

In conclusion, the main results of the paper do not depend on the finiteness of the game. Rather, they depend on the fact that the likelihood of meeting new entrants is sufficiently low from some point in time onward. This property guarantees that there is a period from which point onward it is no longer beneficial to join the sales force regardless of the other agents' behavior and it allows us to treat the game as one with a finite number of periods.

Appendix B: Proofs

Proof of Proposition 1

Assume by way of contradiction that there exists an IC reward scheme R such that $\pi(R) > 0$. Since $q = 0$, no agent purchases the good at a price $\eta^R > 0$. Since $\pi(R) > 0$, it follows that $\phi^R > 0$ and that a subgame perfect Nash equilibrium (SPE) of $\Gamma(R)$ in which the agents purchase distribution licenses exists. Since there is only a finite number of periods, there is a last period $t^\star$ in which one (or more) of the agents is supposed to purchase a license on the equilibrium path of this SPE. By the incentive-compatibility property of R, if $\eta^R = 0$, then $b_\tau^R = 0$ for every $\tau \geq 1$. An agent i who purchases a license in period $t^\star$ cannot expect to sell licenses (or units of the good if $\eta^R > 0$) in this SPE. This is in contradiction to the optimality of purchasing a license at $t^\star$ as i must expect a reward of 0 that does not cover his cost of becoming a distributor $\phi^R + c$.

Proof of Theorem 1

Assume by way of contradiction that there exists an IC one-level scheme R such that $\pi(R) > 0$ and consider an ABEE (σ, β) of $\Gamma(R)$ in which the agents purchase licenses. In an ABEE, from each agent i's perspective, the other agents accept offers (to purchase a license) with probability $\beta\left(M_1^{-i}\right)$ and make offers with probability $\beta\left(M_2^{-i}\right)$ irrespective of the history. Hence, if i finds it optimal to purchase a license in period t following a history h_i^t, then he finds it optimal to do so following every history $\hat{h}_i^t$. In conclusion, the optimality of i's decision to accept or reject an offer depends solely on the period in which it is made. A similar argument shows that, in an ABEE, the optimality of i's decision to make or not to make an offer depends solely on the period in which it is made.

For each $i \in I$, let k_i be the last period in which agent i accepts an offer on the equilibrium path of (σ, β) (if such a period does not exist, set $k_i = 0$). Let $K := \{i \in I : k_i > 0\}$ and $k^\star := max_{i\in K} k_i$. For each $i \in I$, let $\hat{G}_i^t := G_i - \cup_{j \in G_i^t - \{i\}} G_j$. Note that $\hat{G}_i^t \subseteq G_i$ for each $i \in I$ and that $\hat{G}_i^t = G_i$ for every agent $i \notin G^t$ as $G_i^t = \emptyset$. Since for each $t \geq 1$ the players who enter the game prior to period t are equally likely to meet the t-th entrant, it follows that for each $i \in I$, $E[|\hat{G}_i^t| : i \in G^t] = E[|G_{i_t}|]$, where $E[|G_{i_t}|]$ is the expected number of agents in the subtree of G rooted at the t-th entrant. Bhattacharya and Gastwirth (1984, p. 531) show that $E[|G_{i_t}|] = \frac{n-t}{t+1} + 1$. For each $i \in K$ and $t \geq 1$, let v_i^t (respectively, $\tilde{v}_i^t$) denote the expected (respectively, analogy-based expected[14]) number of offers that i makes *to the members of* $\hat{G}_i^t$ if he behaves according to his ABEE strategy and holds a license at the end of period t. Note that $v_i^t \leq \frac{n-t}{t+1}$ (as i cannot make an offer to himself) and set $v := min_{i \in K} v_i^{k^\star}$.

Observe that each $i \in I$ believes that every offer that he makes after period $k^\star$ will be rejected with probability $1 - \beta\left(M_1^{-i}\right)$, whereas every such offer is in fact rejected with probability 1. When an agent j such that $\psi_j = 1$ rejects (respectively, accepts) i's offer, the latter will (respectively, will not) be able to make additional offers to j's successors. Since i's analogy-based expectations underestimate the likelihood that the offers he makes after period $k^\star$ will be rejected, it follows that $\tilde{v}_i^t \leq v_i^t$ for each $t \geq k^\star$.

We now obtain an upper bound for the agents' analogy-based expectations. Note that every offer that is accepted in each period $t \in \{1, ..., k^\star\}$ results in a distributor $l \in D_{k^\star+1}$ who makes, in expectation, $v_l^{k^\star} \geq v$ offers to the members of $\hat{G}_l^{k^\star}$ after period $k^\star$. Since all of these late offers are rejected, the proportion of offers accepted by members of I cannot exceed $\frac{1}{1+v}$. For each $j \in I$, $\beta\left(M_1^{-j}\right)$ is the proportion of offers accepted by members of $I - \{j\}$. In order to obtain an upper bound for $\beta\left(M_1^{-j}\right)$, we subtract the offers that are rejected by agent j after period $k^\star$ from $\left(v_i^{k^\star}\right)_{i \in K}$ (since we are looking for an upper bound, we do not subtract offers that are accepted by j).

There are two types of realization of the order of entry to consider. In the first one, agent j is one of the first $k^\star$ entrants and, therefore, he does not receive any offer after period $k^\star$. Hence, there are no subtractions and the proportion of offers accepted by the members of $I - \{j\}$ in this type of realization is at most $\frac{1}{1+v}$.

In the second type of realization, j is one of the last $n - k^\star$ entrants. For each $t > k^\star$, conditional on $j \notin G^{k^\star}$, there is a probability $\frac{1}{n-k^\star}$ that agent j will enter the game in period t. Note that the ABEE behavior and identity of the members of

[14]Assuming that the other agents accept offers with probability $\beta\left(M_1^{-i}\right)$.

$D_{k^\star+1}$ conditional on j entering the game in period $t > k^\star$ are identical to their ABEE behavior and identity conditional on j entering the game in period $t' > k^\star$, for every $t, t' > k^\star$. Hence, conditional on entering the game after period $k^\star$, agent j receives, in expectation, $\frac{1}{n-k^\star}$ of the offers that are made after period $k^\star$. Thus, the proportion of offers accepted by the members of $I - \{j\}$ in this type of realization is at most $\frac{1}{1+v-\frac{1}{n-k^\star}v} \geq \frac{1}{1+v}$ and $\beta\left(M_1^{-j}\right) \leq \frac{1}{1+v-\frac{1}{n-k^\star}v}$ for each $j \in I$.

Consider an agent $i \in argmin_{j\in K} v_j^{k^\star}$ and assume that $i \in D_{t+1}$, where $t \geq k^\star$. At the end of period t, i believes that, in expectation, he will sell $\beta\left(M_1^{-i}\right)\tilde{v}_i^t$ licenses to members of $\hat{G}_i^t$. Recall that $\tilde{v}_i^t \leq v_i^t$. As $t \geq k^\star$, it follows that $v_i^t \leq v_i^{k^\star}$. Since $i \in argmin_{j\in K} v_j^{k^\star}$, it follows that $v_i^{k^\star} = v \leq \frac{n-k^\star}{k^\star+1}$ and, therefore, $\beta\left(M_1^{-i}\right)\tilde{v}_i^t \leq \frac{n-k^\star}{n} < 1$.

Consider again an agent $i \in argmin_{j\in K} v_j^{k^\star}$ who holds a license and interacts with an agent j in period $t \geq k^\star$. He can sell a license to j. Distributor i believes that by not selling a license to j and, instead, by making offers to j's successors (i.e., the members of $\hat{G}_j^t = G_j$) according to σ_i, in expectation, he will sell $p\beta\left(M_1^{-i}\right)\tilde{v}_i^t \leq p\frac{n-k^\star}{n} < 1$ licenses.[15] Since R is a one-level scheme, i's reward depends only on the number of licenses he sells. Since $p\beta\left(M_1^{-i}\right)\tilde{v}_i^t < 1$, he makes an offer to j and every agent whom he interacts with from period $k^\star$ onward. As $i \in argmin_{j\in K} v_j^{k^\star}$, it follows that every $j \in K$ makes an offer to every agent whom he interacts with after period $k^\star$ such that $v_j^{k^\star} = v$ for every agent $j \in K$.

Consider an agent i such that $k_i = k^\star$. He believes that, in expectation, he will sell $\beta\left(M_1^{-i}\right)\tilde{v}_i^{k^\star} \leq \beta\left(M_1^{-i}\right)v_i^{k^\star} = \beta\left(M_1^{-i}\right)v \leq \frac{n-k^\star}{n}$ licenses if he purchases a license in period $t = k^\star$. Since R is a one-level scheme, i "analogy-based expects" rewards of at most $\frac{n-k^\star}{n}a_1^R - \phi^R$ based on selling licenses. If R is IC, then $a_1^R \leq \phi^R$ (and, if $b_1^R > 0$, i cannot expect any commissions based on selling the good), which is in contradiction to the assumption that there is an agent $i \in K$ who purchases a license in period $k^\star$ in this ABEE.

Proof of Theorem 2

The proof consists of three parts. First, we need to show that if both $p = 0$ and n is sufficiently large, then a pyramid scam can be sustained (this part follows directly from a more general result given in Proposition 2 and, therefore, its proof is omitted). The second part of the proof establishes that it is impossible to sustain a pyramid scam when $p = 1$ (for any value of n). In the third part of the proof, we show that the ability

[15] Notice that we were able to substitute between $\hat{G}_i^t$ and $\hat{G}_j^t$ since they are symmetric (i.e., the probability that the t'-th entrant joins the λ-th level of $\hat{G}_j^t$ is equal to the respective probability that he joins the λ-th level of $\hat{G}_i^t$ for every $t' > t$ and λ).

to sustain a pyramid scam is monotone in p (i.e., if the SO can sustain a pyramid scam given n and p, then he can do so given n and $p' < p$). The second and third parts of the proof will rely on the following lemma, which generalizes the arguments made in the proof of Theorem 1 to multilevel reward schemes.

Lemma 1 *Consider a scheme R and an ABEE of $\Gamma(R)$ in which each agent rejects every offer made after some period $k^\star \leq n-1$. In every such ABEE, in each period $t > k^\star$, every distributor who interacts with an agent makes an offer to the latter.*

Proof. Repeat the first seven paragraphs of the proof of Theorem 1 and consider again an agent $i \in argmin_{j\in K} v_j^{k^\star}$ who holds a license and interacts with the t-th entrant in period $t \geq k^\star$. Denote by $l_{\lambda,t}$ the number of distributors at the λ-th level of the t-th entrant's downline in the case where the latter purchases a license from i, according to i's analogy-based expectations (i.e., assuming that after period t every distributor makes offers with probability $\beta\left(M_2^{-i}\right)$ and agents accept offers with probability $\beta\left(M_1^{-i}\right)$). Set $l_{0,t} := 1$ for each $t \geq k^\star$. Since the likelihood of meeting the new entrant goes down as time progresses, $l_{\lambda,t}$ is weakly decreasing in t.

According to i's analogy-based expectations, selling a license to the t-th entrant raises the expected number of distributors at the λ-th level of i's downline by $l_{\lambda-1,t}$. As we showed in Theorem 1, distributor i cannot believe that, in expectation, he will sell more than $p\beta\left(M_1^{-i}\right)\tilde{v}_i^t \leq p\frac{n-k^\star}{n} < 1$ licenses to the t-th entrant's successors if he does not sell a license to the latter. Since $l_{\lambda,t}$ is weakly decreasing in t, according to i's analogy-based expectations, not selling a license to the t-th entrant and, instead, approaching his successors raises the expected number of distributors at the λ-th level of i's downline by no more than $p\beta\left(M_1^{-i}\right)\tilde{v}_i^t l_{\lambda-1,t}$. Thus, making an offer to the t-th entrant results in a larger "analogy-based expected" number of distributors *at every level* of i's downline compared to not making him an offer. It follows that i makes an offer to every agent whom he interacts with from period $k^\star$ onward. Since $i \in argmin_{j\in K} v_j^{k^\star}$, every $j \in K$ makes an offer to every agent whom he interacts with after period $k^\star$. ■

Set $p = 1$ and assume that there exists an ABEE (σ, β) in which agents purchase licenses on the equilibrium path. We use the same notation from the proofs of Theorem 1 and Lemma 1. By Lemma 1, under σ, each distributor makes an offer to every agent whom he interacts with after period $k^\star$. Since $p = 1$ and every offer that is made after period $k^\star$ is rejected, each $i \in D_{k^\star+1}$ makes an offer to each member of $\hat{G}_i^{k^\star} - \{i\}$. Note that $E[|\hat{G}_i^{k^\star}| : i \in D_{k^\star+1}] = E[|\hat{G}_i^{k^\star}| : i \in G^{k^\star}] = E[|G_{i_{k^\star}}|] = \frac{n-k^\star}{k^\star+1} + 1$. Thus, $v = \frac{n-k^\star}{k^\star+1}$.

Observe that each agent i's downline is a subset of his successors $G_i - \{i\}$ and recall that $E[|G_{i_{k^\star}} - \{i_{k^\star}\}|] = \frac{n-k^\star}{k^\star+1}$. Hence, if the $k^\star$-th entrant purchases a license, he cannot believe that, in expectation, he will have more than $\beta\left(M_1^{-i_{k^\star}}\right)\frac{n-k^\star}{k^\star+1}$ distributors in his downline at the end of the game. In the proof of Theorem 1, we found that $\beta\left(M_1^{-i}\right)\leq\frac{1}{1+v\frac{n-k^\star-1}{n-k^\star}}$ for each $i\in K$. Since $v=\frac{n-k^\star}{k^\star+1}$, it follows that $\beta\left(M_1^{-i_{k^\star}}\right)\frac{n-k^\star}{k^\star+1}\leq\frac{n-k^\star}{n}<1$.

If R is IC, then $a_\tau^R\leq\phi^R$ for every $\tau\geq 1$ (and, as $q=0$, the agents' cannot expect strictly positive sales commissions). Hence, the $k^\star$-th entrant cannot analogy-based expect rewards greater than $\frac{n-k^\star}{n}\phi^R-\phi^R<0$ if he purchases a license. This is in contradiction to the existence of an ABEE in which an agent purchases a license in period $k^\star$.

The last part of the proof shows that the SO's ability to sustain a pyramid scam is monotone in p. Fix $n^\star$ and $p^\star$, and suppose that there exists an IC scheme R such that $\pi(R)>0$. Let (σ,β) be an ABEE of $\Gamma(R)$ in which the agents purchase licenses and define v_i^t, v, $k^\star$, K, $\hat{G}_i^t$, and k_i as they are defined above. By Lemma 1, each distributor makes an offer to every agent whom he interacts with after period $k^\star$ in (σ,β). Hence, $v_i^{k^\star}=v$ for each $i\in K$.

In the proof of Theorem 1, we showed that $\beta\left(M_1^{-i}\right)\leq\frac{1}{1+\frac{n-k^\star-1}{n-k^\star}v}$ for each agent $i\in K$. We now use the fact that the agents' behavior after period $k^\star$ is symmetric in order to obtain a tighter upper bound for the analogy-based expectations of some of the agents. We will later use this bound to show that if $n=n^\star$, then for every $p\leq p^\star$ there exists an IC scheme R'_p such that $\pi\left(R'_p\right)>0$.

Lemma 2 *In the ABEE (σ,β), for each agent $j\in argmax_{i\in K}k_i$ it holds that $\beta\left(M_1^{-j}\right)\leq\frac{1}{1+v}$.*

Proof. Consider (σ,β) and an agent j such that $j\in argmax_{i\in K}k_i$. Let us examine the offers that are made in periods $\{1,...,k^\star\}$ and show that, in expectation, for each such offer that is accepted by a member of $I-\{j\}$, there are at least v offers that are made to (and rejected by) members of $I-\{j\}$ after period $k^\star$.

First, consider agent j. Every offer that he makes is received by an agent $l\in I-\{j\}$. If an agent $l\in I-\{j\}$ accepts j's offer in a period $t\leq k^\star$, then l makes, in expectation, $v_l^{k^\star}=v$ offers to the members of $\hat{G}_l^{k^\star}\subseteq G_l$ after period $k^\star$. All of the latter offers are made to (and rejected by) members of $I-\{j\}$ as $j\notin G_l$.

Second, consider a player (i.e., an agent or the SO) $i\neq j$ who, in this ABEE, makes an offer if he interacts with an agent in period $t\leq k^\star$. There are two types of realization (of the order of entry) that we need to consider. In the first type, agent j is one of the first $t-1$ entrants. Every offer that is accepted by $l\in I-\{j\}$ in period t results in a distributor who makes, in expectation, $v_l^{k^\star}=v$ offers to the members of $\hat{G}_l^{k^\star}\subseteq G_l$ after period $k^\star$. Again, all of these offers are made to (and rejected by) members of $I - \{j\}$ as $j \notin G_l$.

In the second type of realization, agent j is one of the last $n - t + 1$ entrants (i.e., $j \notin G^{t-1}$). Note that $Pr\left(j \text{ } is \text{ } the \text{ } t'\text{-}th \text{ } entrant | j \notin G^{t-1}\right) = \frac{1}{n-t+1}$ for each $t' \geq t$. Thus, in a fraction of $\frac{1}{n-t+1}$ of these realizations, player i makes the period-t offer to agent j and in a fraction $\frac{n-t}{n-t+1}$ of these realizations he makes the period-t offer to an agent $l \neq j$. Conditional on accepting the offer, $l \neq j$ (respectively, j) makes, in expectation, v offers to the members of $\hat{G}_l^{k^\star}$ (respectively, $\hat{G}_j^{k^\star}$). Note that j always accepts a period-t offer since $j \in argmax_{i \in K} k_i$ and the likelihood of meeting the new entrant goes down as time progresses. Since $Pr\left(j \text{ } is \text{ } the \text{ } t'\text{-}th \text{ } entrant | j \notin G^{t}\right) = \frac{1}{n-t}$ for each $t' > t$, conditional on an agent $l \neq j$ accepting the period-t offer, in expectation, agent j is the recipient of $\frac{1}{n-t}$ of l's offers after period $k^\star$. In conclusion, in this type of realization, a fraction $\frac{y(n-t)}{n-t+1}$ of player i's period-t offers are accepted by members of $I - \{j\}$ (where $y < 1$ if some members of $I - \{j\}$ reject the period-t offer) and a fraction $\frac{y(n-t)}{n-t+1} + \frac{1}{n-t+1}$ of player i's period-t offers are accepted by members of I. These accepted offers result in distributors who make, in expectation, at least $\frac{y(n-t)}{n-t+1}\left(1 - \frac{1}{n-t}\right)v + \frac{1}{n-t+1}v \geq \frac{y(n-t)}{n-t+1}v$ offers to members of $I - \{j\}$ after period $k^\star$.

In all of the above cases (offers by j, offers by $i \neq j$ when j enters the game before period t, and offers by $i \neq j$ when j enters the game in a period $t' \geq t$), for every offer that is accepted by a member of $I - \{j\}$ in period $t \leq k^\star$, in expectation, there are at least v offers that are made to members of $I - \{j\}$ after period $k^\star$. Since all of these late offers are rejected, it follows that $\beta\left(M_1^{-j}\right) \leq \frac{1}{1+v}$. ■

Let $\mathcal{M} = \left\{R' | a_\tau^{R'} = x\phi^{R'} \text{ } and \text{ } b_\tau^{R'} = 0 \text{ } for \text{ } every \text{ } \tau \geq 1, x \leq 1, \phi^{R'} = \phi^R\right\}$ be a set of IC schemes. We now examine a profile σ', and use the upper bound from Lemma 2 in order to show that, for every $p \leq p^\star$, there exists a scheme $R'_p \in \mathcal{M}$ such that σ' is part of an ABEE (σ', β') of $\Gamma\left(R'_p\right)$ in which the SO's expected profit is strictly positive.

Let σ' be a profile in which the SO makes an offer to every agent whom he interacts with in period $k^\star$ (i.e., the last period in which agents accept offers in (σ, β)) and never makes offers in any period $t > k^\star$. Under σ', each distributor makes an offer to every agent whom he interacts with, and each agent accepts every offer he receives in each period $t \leq k^\star$ and rejects every offer he receives after period $k^\star$. Note that we do not make any assumption about the SO's behavior prior to $k^\star$.

Let $p = p^\star$. We now examine the analogy-based expectations that are induced by σ' and $p^\star$. Consider an agent i who holds a license at the end of period $k^\star$. According to σ', agent i will make an offer to every member of $\hat{G}_i^{k^\star}$ whom he will interact with and will not interact with any member of $G_i - \hat{G}_i^{k^\star}$ after period $k^\star$ (as there is a distributor on the path between i and any agent who joins $G_i - \hat{G}_i^{k^\star}$ after period $k^\star$). By Lemma

1, in expectation, after period $k^\star$, agent i makes the same number of offers as an agent j who purchases a license in period $k^\star$ in the ABEE (σ, β). Thus, under σ', every offer that is made in period $t \leq k^\star$ results in a distributor who makes, in expectation, v offers after period $k^\star$. Since no offers are rejected prior to period $k^\star + 1$ according to σ', the expected proportion of accepted offers is exactly $\frac{1}{1+v}$. Since the agents' behavior under σ' is symmetric, it follows that for each agent i, $\beta'\left(M_1^{-i}\right) = \frac{1}{1+v}$ is consistent with σ'. It is easy to see that for each $i \in I$, $\beta'\left(M_2^{-i}\right) = 1$ is consistent with σ'.

Consider (σ', β') and a scheme $R' \in \mathcal{M}$. By making an offer only in period $k^\star$, the SO makes a strictly positive expected profit. The SO is indifferent between making an offer and not doing so in every period $t > k^\star$. Each distributor finds it optimal to make an offer to each agent whom he interacts with as one's payoff under R' depends solely on the number of distributors in his downline and $\beta'\left(M_2^{-i}\right) = 1$. In order to show that (σ', β') is an ABEE of $\Gamma(R')$, it remains to verify the optimality of the agents' decisions to accept or reject offers.

Let us examine the optimality of the $k^\star$-th entrant's decision to purchase a license in (σ', β'). We wish to show that for some $R' \in \mathcal{M}$, he finds it optimal to purchase a license. We shall use the fact that in the ABEE (σ, β) of $\Gamma(R)$ there is an agent i who accepts offers in period $k^\star$. By the incentive compatibility of R, a scheme $R' \in \mathcal{M}$ with $x = 1$ provides commissions weakly greater than the ones provided by R for every level of i's downline. Hence, if we fix $\beta\left(M_1^{-i}\right)$ and $\beta\left(M_2^{-i}\right)$, and change the scheme from R to $R' \in \mathcal{M}$ for which $x = 1$, then i must still find it optimal to purchase a license at time $k^\star$. If, in addition, we raise $\beta\left(M_1^{-i}\right)$ to $\frac{1}{1+v}$ (by Lemma 2, $\beta\left(M_1^{-i}\right) \leq \frac{1}{1+v}$) and $\beta\left(M_2^{-i}\right)$ to 1 as induced by (σ', β'), then we only increase i's analogy-based expected payoff. Hence, if i finds it optimal to purchase a license in period $k^\star$ given $R, \beta\left(M_1^{-i}\right)$, and $\beta\left(M_2^{-i}\right)$, then there exists a scheme $R' \in \mathcal{M}$ such that every agent $i \in I$ finds it optimal to purchase a license in period $k^\star$ given $\beta'\left(M_1^{-i}\right), \beta'\left(M_2^{-i}\right)$, and R'.

Since the likelihood of meeting the new entrant goes down as time progresses, it is easy to set $x \leq 1$ such that each $i \in I$ finds it suboptimal to purchase a license in each period $t > k^\star$ and each $i \in I$ finds it optimal to purchase a license in each period $t \leq k^\star$. In conclusion, if $\pi(R) > 0$, then for some $R' \in \mathcal{M}$ the profile σ' is part of an ABEE of $\Gamma(R')$ in which the SO makes a strictly positive expected profit.

Set $p < p^\star$. To complete the proof, we show that it is possible to find a scheme $R'_p \in \mathcal{M}$ such that σ' is part of an ABEE of $\Gamma\left(R'_p\right)$. Clearly, given any reward scheme $R'_p \in \mathcal{M}$, the SO's behavior under σ' induces a strictly positive expected profit of at least $\frac{1}{k^\star}\phi^{R'_p}$ and each distributor finds it optimal (w.r.t. his induced analogy-based expectations) to make an offer to every agent whom he interacts with.

To finish the proof, it is left to show that there is a scheme $R'_p \in \mathcal{M}$ such that, given the analogy-based expectations that are induced by σ' and p, the $k^{\star}$-th entrant finds it optimal to purchase a license in $\Gamma\left(R'_p\right)$. Note that under each scheme $R'' \in \mathcal{M}$, each distributor's reward is the number of agents in his downline multiplied by x and that, given the analogy-based expectations that are induced by σ' and $p^{\star} > p$, the $k^{\star}$-th entrant finds it optimal to purchase a license. All we need to show now is that the expected number of distributors in the downline of a distributor who purchases a license in period $k^{\star}$, according to his analogy-based expectations that are induced by σ' and p, is weakly decreasing in p. It will then be easy to set $x \leq 1$ such that each agent finds it optimal to purchase a license in each period $t \leq k^{\star}$ and suboptimal to purchase a license in each period $t > k^{\star}$ of the scheme's induced game.

Let $l_z = E[|\{j \in G_{i_{k^{\star}}} : d_G(i_{k^{\star}}, j) = z\}|]$ be the expected number of agents in the z-th level of the subtree of G rooted at the $k^{\star}$-th entrant. For example, $l_1 = \sum_{j=k^{\star}+1}^{n} \frac{1}{j}$ and $l_2 = \sum_{j_1=k^{\star}+1}^{n-1} \sum_{j_2=j_1+1}^{n} \frac{1}{j_1 j_2}$. Denote by $v(p)$ the expected number of offers that the $k^{\star}$-th entrant makes given σ' (recall that, by Lemma 1, he makes an offer to every agent whom he interacts with after period $k^{\star}$ and, therefore, $v(p) = l_1 + p l_2 + p^2 l_3 + \ldots$). The number of distributors in the $k^{\star}$-th entrant's downline according to p, σ' and their induced analogy-based expectations is

$$\frac{l_1}{(1+v(p))} + \frac{l_2(1+pv(p))}{(1+v(p))^2} + \frac{l_3(1+pv(p))^2}{(1+v(p))^3} + \frac{l_4(1+pv(p))^3}{(1+v(p))^4} + \ldots \tag{2}$$

Let us show that the derivative of (2) w.r.t. p is less than or equal to 0 for $p \in [0,1]$. Derive the z-th component of (2), given by $\frac{l_z(1+pv(p))^{z-1}}{(1+v(p))^z}$, w.r.t. p to obtain

$$\begin{aligned}&\frac{l_z}{(1+v(p))^{2z}}\left[(z-1)(1+v(p))^z(1+pv(p))^{z-2}(v(p)+v'(p)p)\right] \\ &\quad -\frac{l_z}{(1+v(p))^{2z}}\left[z(1+v(p))^{z-1}(1+pv(p))^{z-1}v'(p)\right]\end{aligned} \tag{3}$$

The derivative of (2) w.r.t. p is the summation over (3) for $z = 1, \ldots, n-k^{\star}$. It can be written as $\sum_{z=1}^{n-k^{\star}} \frac{z(1+pv(p))^{z-1}}{(1+v(p))^{z+1}}\left[l_{z+1}(v(p)+v'(p)p) - l_z v'(p)\right]$ since $l_{n-k^{\star}+1} = 0$. The latter expression equals the sum of the elements of the following square matrix:

$$\begin{pmatrix} (l_2l_1 - l_1l_2)\,\gamma_{11} & (l_2l_2 - l_1l_3)\,\gamma_{12} & (l_2l_3 - l_1l_4)\,\gamma_{13} & \dots & (l_2l_{n-k^\star} - 0)\,\gamma_{1(n-k^\star)} \\ (l_3l_1 - l_2l_2)\,\gamma_{21} & 0 \times \gamma_{22} & (l_3l_3 - l_2l_4)\,\gamma_{23} & \dots & \vdots \\ (l_4l_1 - l_3l_2)\,\gamma_{31} & (l_4l_2 - l_3l_3)\,\gamma_{32} & 0 \times \gamma_{33} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0 - l_{n-k^\star}l_2)\,\gamma_{(n-k^\star)1} & \dots & \dots & \dots & 0 \end{pmatrix}$$

where $\gamma_{xy} = \frac{xyp^{y-1}(1+pv(p))^{x-1}}{(1+v(p))^{x+1}}$. Observe that $\gamma_{xy} \geq \gamma_{yx}$ if and only if $x \geq y$. Hence, the sum of the matrix's cells is less than or equal to 0 if $\frac{l_2}{l_1} \geq \frac{l_3}{l_2} \geq ... \geq \frac{l_{n-k^\star}}{l_{n-k^\star-1}}$.

To prove the above inequality, consider the subtree of G rooted at the $k^\star$-th entrant and denote by $o_{\lambda,t}$ the expected number of agents at the λ-th level of that subtree at the end of period $t > k^\star$. An increase in t strictly raises $\frac{o_{2,t}}{o_{1,t}}$ since $o_{1,t} - o_{1,t-1} = \frac{1}{t}$ and $o_{2,t} - o_{2,t-1} = \frac{1}{t}\left(\frac{1}{k^\star+1} + ... + \frac{1}{t-1}\right)$. Observe that

$$\frac{o_{\lambda+1,t}}{o_{\lambda,t}} = \frac{o_{\lambda+1,t-1} + \frac{1}{t}o_{\lambda,t-1}}{o_{\lambda,t-1} + \frac{1}{t}o_{\lambda-1,t-1}} \tag{4}$$

The LHS of (4) is weakly increasing in t when it is well defined (i.e., when $\lambda \leq t - k^\star$) since $\frac{o_{2,t}}{o_{1,t}}$ is increasing in t. Extend the branching process to $n+1$ periods, set $t = n+1$, and let $\lambda \leq n - k^\star$. Since the LHS of (4) is weakly increasing in t, it must be that $\frac{o_{\lambda+1,n+1}}{o_{\lambda,n+1}} \geq \frac{o_{\lambda+1,n}}{o_{\lambda,n}}$. By the equality in (4), $\frac{o_{\lambda,n}}{o_{\lambda-1,n}} \geq \frac{o_{\lambda+1,n+1}}{o_{\lambda,n+1}} \geq \frac{o_{\lambda+1,n}}{o_{\lambda,n}}$. Since $\frac{o_{\lambda,n}}{o_{\lambda-1,n}} = \frac{l_\lambda}{l_{\lambda-1}}$, it follows that $\frac{l_2}{l_1} \geq ... \geq \frac{l_{n-k^\star}}{l_{n-k^\star-1}} \geq \frac{l_{n-k^\star+1}}{l_{n-k^\star}} = 0$.

Proof of Theorem 3

The proof of Theorem 3 is omitted as it follows directly from Proposition 2 for $\alpha = 1$.

Proof of Proposition 2

Let $\mathcal{M} = \left\{R|\phi^R > 0, a_1^R = a_2^R = x\phi^R, x \leq 1, a_\tau^R = b_\tau^R = 0 \text{ } \textit{for every } \tau > 2\right\}$ be a set of IC two-level reward schemes, fix an arbitrary $\alpha \in (0,1]$, and set $p = 0$. Consider an arbitrary scheme $R \in \mathcal{M}$ and a profile σ such that: every analogy-based reasoner accepts (respectively, every rational agent rejects) every offer he receives in period $t = 1$ and every agent rejects every offer that he receives in each period $t > 1$, the SO makes an offer only in period $t = 1$, and every distributor makes an offer to every agent whom he meets. Note that σ guarantees the SO an expected profit of $\frac{\lceil \alpha n \rceil}{n}\phi^R$.

Let us examine the players' behavior under σ. First, since $p = 0$, making an offer is always optimal for a distributor who meets an agent. Second, when the SO meets an agent, the former finds it optimal to offer a license at $t=1$ and is indifferent between making an offer and not doing so at each $t>1$. It is suboptimal for every fully rational agent to accept offers as such agents correctly believe that they will not be able to sell licenses. It is left to show that every analogy-based reasoner finds it optimal to accept the SO's offer at $t=1$ and reject offers at each $t>1$.

Consider the analogy-based expectations of an analogy-based reasoner i. Under σ, the SO makes an offer at $t=1$. This offer is accepted by analogy-based reasoners and rejected by rational agents. If this offer is accepted by an analogy-based reasoner j, then, in expectation, he makes $\sum_{t=2}^{n}\frac{1}{t}$ additional offers that are all rejected (as he is expected to meet $\sum_{t=2}^{n}\frac{1}{t}$ agents from period 2 onward). If $j\neq i$, then $\frac{1}{n-1}$ of these offers are received and rejected by i. Thus,

$$\beta\left(M_1^{-i}\right)=\frac{\frac{\lceil\alpha n\rceil}{n}-\frac{1}{n}}{1-\frac{1}{n}+\left(\frac{\lceil\alpha n\rceil}{n}-\frac{1}{n}\right)\frac{n-2}{n-1}\sum_{t=2}^{n}\frac{1}{t}+\frac{1}{n}\sum_{t=2}^{n}\frac{1}{t}}$$

Note that the subtraction of $\frac{1}{n}$ is due to the fact that i's analogy-based expectations do not depend on offers that he receives. Since $p=0$, each analogy-based reasoner i who purchases a license in period 1 correctly believes that he will make $\sum_{t=2}^{n}\frac{1}{t}$ offers from $t=2$ onward. It follows that each analogy-based reasoner believes that he will sell $\beta\left(M_1^{-i}\right)\sum_{t=2}^{n}\frac{1}{t}$ licenses if he purchases a license. Since the harmonic sum diverges, this expression approaches 1 as n goes to infinity.

What is the expected number of distributors in the second level of the downline of an analogy-based reasoner i who purchases a license in period 1 according to his analogy-based expectations? It is the expected number of agents at the second level of G_i multiplied by i's analogy-based expectations that two agents (one at the first level of his downline and another at the second level of his downline) will accept an offer: $\beta\left(M_1^{-i}\right)^2\sum_{j_1=2}^{n-1}\sum_{j_2=j_1+1}^{n}\frac{1}{j_1 j_2}$. When n goes to infinity, it approaches

$$\frac{\alpha^2\sum_{j_1=2}^{n-1}\sum_{j_2=j_1+1}^{n}\frac{1}{j_1 j_2}}{1+2\alpha\sum_{j=2}^{n}\frac{1}{j}+\alpha^2\left(\sum_{j=2}^{n}\frac{1}{j}\right)^2}=\frac{\alpha^2\sum_{j_1=2}^{n-1}\sum_{j_2=j_1+1}^{n}\frac{1}{j_1 j_2}}{1+2\alpha\sum_{j=2}^{n}\frac{1}{j}+\alpha^2\left(2\sum_{j_1=2}^{n-1}\sum_{j_2=j_1+1}^{n}\frac{1}{j_1 j_2}+\sum_{j=2}^{n}\frac{1}{j^2}\right)}$$

This expression approaches $\frac{1}{2}$ as n goes to infinity. Thus, for sufficiently large n, analogy-based reasoner i expects rewards arbitrarily close to $\frac{3x\phi^R}{2}-\phi^R$ if he purchases a license in period 1. Since the likelihood of meeting the new entrant goes down as time progresses, i expects a smaller reward if he purchases a license in period $t>1$. It is possible to choose $R'\in\mathcal{M}$ such that it is optimal for i and every other analogy-based reasoner to purchase a license at $t = 1$ (since $\frac{3x\phi^{R'}}{2} - \phi^{R'} > c$ for some $x \leq 1$ and $\phi^{R'}$) and it is not optimal to do so at any $t > 1$.

Proof of Theorem 4

The first part of the proof shows that for any reward scheme R there exists an IC reward scheme $R^\star$ such that $\phi^{R^\star} = 0$, $a_\tau^{R^\star} = 0$ for every $\tau \geq 1$, and $\pi(R^\star) \geq \pi(R)$. In the second part of the proof, we show that a profit-maximizing scheme exists.

Part 1. Consider a scheme R and denote by σ an SPE of $\Gamma(R)$ in which the agents purchase licenses (if such an SPE does not exist, then $\pi(R') \geq \pi(R)$ for any scheme R' as the SO can always refrain from selling licenses). Let $k^\star$ be the last period in which any agent purchases a license in σ. For every $t \geq 1$, denote by z_t the expected number of agents whom the t-th entrant will interact with (i.e., agents who may purchase the good from him) if he purchases a license but does not sell licenses after period t. Observe that $z_{k^\star} q b_1^R \geq c + \phi^R$ as the $k^\star$-th entrant does not expect to sell licenses in σ. Note that under σ, the t-th entrant obtains, in expectation, commissions of at least $z_t q b_1^R$ if he purchases a license. Since the likelihood of meeting the new entrant goes down as time progresses, for every $t < k^\star$ it holds that $z_t > z_{k^\star}$ and $z_t q b_1^R > z_{k^\star} q b_1^R$. Finally, note that if $z_{k^\star} q b_1^R > c + \phi^R$, then the SO can increase his profit by increasing ϕ^R. Hence, we can assume without loss of generality that $z_{k^\star} q b_1^R = c + \phi^R$.

Consider a scheme $R^\star$ such that $\eta^{R^\star} = \eta^R$, $z_{k^\star} q b_1^{R^\star} = c$, $\phi^{R^\star} = 0$, and for every $\tau \geq 1$, it holds that $a_\tau^{R^\star} = 0$ and $b_\tau^{R^\star} = b_1^{R^\star} p^{\tau-1}$. Note that under $R^\star$, each distributor j obtains an expected transfer of $qp^{x-y-1} b_{y+1}^{R^\star} = qp^{x-1} b_1^{R^\star}$ for every potential sale to an agent i such that $d_G(j,i) = x$ and there are y distributors on the path connecting j and i. Since this expression is independent of y, each distributor j who interacts with an agent is indifferent between selling a license to him and not doing so regardless of j's beliefs about future play. It follows that, for each $t \geq 1$, in every SPE of $\Gamma(R^\star)$, if the t-th entrant purchases a license, then, in expectation, he will obtain commissions of $z_t q b_1^{R^\star}$ (i.e., as he would if he were not selling licenses.).

Since $z_{k^\star} q b_1^{R^\star} = c = z_{k^\star} q b_1^R - \phi^R$ and $z_{k^\star} < z_t$ for $t < k^\star$, in expectation, the SO's net transfers (i.e., commissions minus entry fees) to a distributor who purchases a license in period $t \leq k^\star$ are (weakly) smaller under $R^\star$ than they are under R given that σ is played. Hence, if σ is played in $\Gamma(R^\star)$, then the SO's expected profit will be higher than it is when σ is played in $\Gamma(R)$.

Since $z_{k^\star} q b_1^{R^\star} = c$, every agent who enters $\Gamma(R^\star)$ in each period $t \leq k^\star$ (respectively, $t \geq k^\star$) finds it optimal (respectively, suboptimal) to purchase a license in every SPE of $\Gamma(R^\star)$. We showed above that each distributor who interacts with an agent is indifferent between selling a license to him and not doing so. Hence, the agents' strategies from σ (i.e., σ_{-SO}) are part of an SPE[16] in $\Gamma(R^{\star})$.

If $R^{\star}$ is IC, then the first part of the proof is completed. If $R^{\star}$ is not IC, then $b_1^{R^{\star}} > \eta^{R^{\star}}$. It follows that $\pi(R^{\star})$ is obtained only when the SO does not sell any licenses. Thus, $\pi(R)$ is obtained only when the SO does not sell any licenses. Hence, $\pi(R') \geq \pi(R)$ for every IC reward scheme R'.

Part 2. A scheme R is said to be *geometric* if there is a number $s \leq 1$ such that $\frac{b_{\tau+1}^R}{b_\tau^R} = s$ for each $\tau \geq 1$. Let $\mathcal{Q}$ be the set of IC geometric schemes with $s = p$, $b_1^R = \frac{c}{qz_t}$ for some $t \in \{1, ..., n-1\}$, $\phi^R = 0$, $\eta^R = 1$, and $a_\tau^R = 0$ for each $\tau \geq 1$. Part 1 showed that if $\mathcal{Q} \neq \emptyset$, then $sup_{R \in \mathcal{Q}} \pi(R) \geq \pi(R')$ for every scheme R' and if $\mathcal{Q} = \emptyset$, then every scheme is profit maximizing. As $\mathcal{Q}$ is finite, there exists a profit-maximizing scheme.

Proof of Proposition 3

We shall need additional notation for this proof. Let $(R_n)_{n=1}^{\infty}$ be a sequence of IC schemes such that each R_n is profit maximizing when there are n agents. For each $n \in \mathbb{N}$, let (σ^n, β^n) be an ABEE of $\Gamma(R_n)$ that induces an expected profit of $\pi(R_n)$. We use k_n to denote the last period in which the agents accept offers to purchase a license in σ^n. If such a period does not exist, then $k_n = 0$.

For each $t \geq 1$ and distributor $j \in D_t$, if j sells a license to the t-th entrant, then we form a link $j \to i_t$. This induces a *distribution tree* T, rooted at the SO: each node of the tree (besides the root) represents a distributor. Denote the set of rooted directed trees by $\mathbb{T}$ and the tree induced by σ^n by T^n. We shall use $Pr(T^n = T|\sigma^n)$ to denote the probability that σ^n results in the distribution tree T. For each $T \in \mathbb{T}$, the length of the directed path connecting i and j in T is denoted by $d_T(i, j)$.

Let us introduce some basic facts about each ABEE (σ^n, β^n). Since the likelihood of meeting the new entrant goes down as time progresses, every agent accepts every offer that he receives prior to period k_n. Since $p = 0$, every distributor makes an offer to every agent whom he meets and does not interact with agents whom he is not directly linked to. Thus, every $i \in D_{k_n+1}$ makes, in expectation, $\sum_{j=k_n+1}^{n} \frac{1}{j}$ offers after period k_n. Hence, every offer that is accepted at $t \leq k_n$ results in a distributor who makes, in expectation, $\sum_{j=k_n+1}^{n} \frac{1}{j}$ offers after period k_n. Note that all of these late offers are rejected and that no offers are rejected prior to period k_n. The SO does not make offers

[16] So far, we have assumed that an agent who is indifferent between making and not making (respectively, accepting and rejecting) an offer makes (respectively, accepts) it. This assumption selects an SPE of $\Gamma(R^{\star})$ that maximizes the SO's expected profit (not necessarily σ) and has no effect on any of the results.

after period k_n as such offers are rejected and negatively affect the agents' analogy-based expectations. It follows that the proportion of accepted offers is $\frac{1}{1+\sum_{j=k_n+1}^{n}\frac{1}{j}}$. Since the agents' strategies are symmetric, for each $i \in I$, $\beta\left(M_1^{-i}\right) = \frac{1}{1+\sum_{j=k_n+1}^{n}\frac{1}{j}}$. Since each distributor makes an offer to every agent whom he meets, $\beta\left(M_2^{-i}\right) = 1$ for every $i \in I$. We denote these analogy-based expectations by β_1^n and β_2^n, respectively.

Lemmata 3–6 confirm the intuition that the high potential gains from trade make the distributors more valuable to the SO as they show that for large values of n, at the optimum, the number of distributors is arbitrarily large with a probability arbitrarily close to 1. Lemma 3 is the first step in this direction: it focuses on large values of n and provides a lower bound for $\pi\left(R_n\right)$.

Lemma 3 *There exist two numbers $\hat{\gamma} \in (0,1)$ and $n\left(\hat{\gamma}\right) \in \mathbb{N}$ such that $\pi\left(R_n\right) \geq \hat{\gamma}n$ for every $n > n\left(\hat{\gamma}\right)$.*

Proof. Consider a large $n \in \mathbb{N}$ and a small $\gamma > 0$ such that $\frac{c}{q(log(1/\gamma)-1)} < 1$. Set $k = \lceil \gamma n \rceil$. Let R be a scheme such that $\eta^R = 1$, $a_\tau^R = 0$ for every $\tau \geq 1$, $b_1^R = \frac{c}{q\sum_{i=k+1}^{n}\frac{1}{i}}$, $\phi^R = 0$, and $b_\tau^R = 0$ for every $\tau > 1$. By definition, $\pi\left(R_n\right) \geq \pi\left(R\right)$. We shall obtain a lower bound for $\pi\left(R_n\right)$ by describing the SO's expected profit in an ABEE of $\Gamma\left(R\right)$.

Consider a profile of strategies σ in which every agent $i \in I$ accepts (respectively, rejects) every offer that he receives at every time $t \leq k$ (respectively, $t > k$) and, for every $t \geq 1$, every distributor $j \in D_t$ makes an offer if he meets an agent. Let us verify that σ is part of an ABEE. As $p = 0$, every distributor finds it optimal to make an offer at every $t \geq 1$. Since $b_1^R = \frac{c}{q\sum_{i=k+1}^{n}\frac{1}{i}}$ and there are no other commissions, every agent finds it optimal to purchase a distribution license if and only if $t \leq k$ (regardless of his analogy-based expectations). Given σ, if the SO makes an offer to every agent whom he meets, the SO's expected profit is greater than

$$qk[1+\sum_{i=k+1}^{n}\frac{1}{i}]\left(1-b_1^R\right) \tag{5}$$

If n is sufficiently large, then $b_1^R < \frac{c}{q(log(1/\gamma)-1)}$ and $1+\sum_{i=k+1}^{n}\frac{1}{i} > log\left(1/\gamma\right)$. Hence, the SO's expected profit is greater than

$$n\hat{\gamma} := n\gamma q\left(1-\frac{c}{q\left(log\left(1/\gamma\right)-1\right)}\right)log\left(1/\gamma\right) \tag{6}$$

■

In Lemma 4, we use the lower bound in (6) to obtain a lower bound for k_n, the last period in which agents purchase licenses under σ^n.

Lemma 4 *There exist* $\bar{\gamma} \in (0,1)$ *and* $n(\bar{\gamma}) \in \mathbb{N}$ *such that* $k_n > \bar{\gamma} n$ *for every* $n > n(\bar{\gamma})$.

Proof. How large can the SO's expected profit be given a fixed k_n? At best, the SO obtains $1+B$ from each of the first $k_n \geq 1$ entrants and 1 from every agent who purchases the good after period k_n. Hence, the SO's expected profit cannot exceed:

$$k_n (1+B) + (k_n+1) \sum_{i=k_n+1}^{n} \frac{1}{i} < k_n (2+B) + 1 + log(n) + k_n log(n/k_n) \tag{7}$$

Consider $\bar{\gamma} \in (0,1)$ such that $\bar{\gamma}(2+B) < \hat{\gamma}/3$ and $\bar{\gamma} log(1/\bar{\gamma}) < min\{\hat{\gamma}/3, 1/e\}$. There exists $n(\bar{\gamma})$ such that if $n > n(\bar{\gamma})$ and $k_n \leq \bar{\gamma} n$, then the lower bound obtained in (6) is strictly greater than the RHS of (7). It follows that there exists a number $n(\bar{\gamma})$ such that $k_n > \bar{\gamma} n$ for every $n > n(\bar{\gamma})$. ∎

Our objective is to show that for large values of n, the number of distributors that results from σ^n is likely to be large as well. Lemma 4 establishes that k_n is large for large values of n. However, the SO may choose not to make offers in the early periods of $\Gamma(R_n)$ and, therefore, the number of distributors $|T^n|$ can be less than k_n. It can be shown that, under σ^n, the SO makes offers to every agent whom he meets in a set of adjacent periods $\{s_n, s_n+1, ..., k_n\}$. Note that s_n can be strictly greater than 1. In the next lemma, we focus on large values of n and provide an upper bound for s_n, which denotes the first period in which the SO makes an offer under σ^n. The upper bound is independent of n.

Lemma 5 *There exists a number* $n^\star$ *such that* $s_n \leq \frac{2(1+B)}{\hat{\gamma}}$ *for every* $n > n^\star$.

Proof. The SO's expected profit from selling the good in periods $1, ..., s_n$ is $q\eta^{R_n} \sum_{i=1}^{s_n} \frac{1}{i}$ $< qlog(n) + q$. Since $p = 0$, if the SO does not make an offer to agent $i \in I$, then i's successors (i.e., members of $G_i - \{i\}$) will not purchase the good nor will they become distributors. Hence, the set of potential distributors and buyers from period s_n onward is $\hat{G}_{SO}^{s_n-1} = G - \cup_{j \in G^{s_n-1}-\{SO\}} G_j - \{SO\}$. Note that $E[|\hat{G}_{SO}^{s_n-1}|] = E[|G_{i_{s_n-1}}|] - 1 = \frac{n-s_n+1}{s_n}$. Thus, even if every agent who interacts with a distributor or with the SO from period s_n onward purchases a license and the good, the SO's expected profit cannot exceed $qlog(n) + q + \frac{n-s_n+1}{s_n}(1+B) \leq qlog(n) + q + \frac{n}{s_n}(1+B)$. By Lemma 3, for large values of n, it must be that $\frac{n}{s_n}(1+B) \geq \frac{\hat{\gamma} n}{2}$ and $s_n \leq \frac{2(1+B)}{\hat{\gamma}}$. ∎

Lemmata 4 and 5 establish that, for large values of n, the SO starts making offers in period $s_n \leq \frac{2(1+B)}{\hat{\gamma}}$. He, as well as every distributor, makes offers until period $k_n > \bar{\gamma} n$. If $s_n > 1$, then $|T^n|$ is a random variable that can take values in $\{1, ..., k_n - s_n + 2\}$.

Lemma 6 *Fix arbitrary $\epsilon > 0$ and $\rho > 0$. There exists a number $n(\epsilon, \rho)$ such that $Pr(|T^n| > \rho) > 1 - \epsilon$ for every $n > n(\epsilon, \rho)$.*

Proof. Observe that, under σ^n, in every $t \in \{s_n, ..., k_n\}$, every $i \in D_t$ who meets an agent makes him an offer and the latter accepts the offer. Hence, $|T^n|$ corresponds to the number of nodes in a subtree of a uniform random recursive tree of size $k_n + 1$, rooted at the s_n-th node. By Theorem 1 in Athreya and Ney (1972, p. 220), as k_n goes to infinity, the limiting distribution of $\frac{|T^n|}{k_n+1}$ is[17] $Beta(1, s_n - 1)$. By Lemma 4, $k_n > \bar{\gamma} n$ for sufficiently large values of n. By Lemma 5, s_n is bounded from above. Thus, for a sufficiently large n, $Pr(|T^n| > \rho)$ is arbitrarily close to 1. ■

So far, we have established that if n is large, then it is likely that the number of distributors induced by σ^n is large as well. In the second part of the proof, we shall use this finding and solve the SO's "dual" problem: how to compensate the distributors (such that the k_n-th entrant is willing to pay ϕ^{R_n} for a license) in the least costly way given σ^n and ϕ^{R_n}.

Let us calculate *cost-benefit* ratios for every commission. The cost (respectively, benefit) is the increase in the SO's expected cost (respectively, the k_n-th entrant's willingness to pay for a license) when a commission is increased given σ^n and its induced analogy-based expectations.

In order to determine the costs for the different commissions, we shall examine the distribution of distances between the SO and the distributors given σ^n. Let $\mathbb{1}(d_T(SO, j) > \tau) \in \{0, 1\}$ be an indicator that equals 1 if and only if $d_T(SO, j) > \tau$. The next lemma shows that, for large values of n, under σ^n, a proportion arbitrarily close to 1 of the distributors are expected to be at a distance of more than τ from the SO with probability arbitrarily close to 1.

Lemma 7 *Fix arbitrary $\epsilon > 0$ and $\tau \geq 1$. There exists a number $n(\epsilon, \tau)$ such that for every $n > n(\epsilon, \tau)$,*

$$\sum_{T \in \mathbb{T}} \sum_{j \in T} Pr(T^n = T : \sigma^n) \mathbb{1}(d_T(SO, j) > \tau) > (1 - \epsilon) \sum_{T \in \mathbb{T}} \sum_{j \in T} Pr(T^n = T : \sigma^n) \quad (8)$$

Proof. The distribution tree that is induced by σ^n is a uniform random recursive tree rooted at the SO. Denote the i-th agent to buy a license by j. The distance $d_T(SO, j)$ corresponds to the distance between the root and the $(i+1)$-th node of a uniform random recursive tree. This distance is independent of the size of the tree and is referred to as the *insertion depth* of the $(i+1)$-th node. A central limit theorem

[17] Theorem 5.1 in Mahmoud and Smythe (1991) presents this result in terms of recursive trees.

for random recursive trees (Theorem 1 in Mahmoud, 1991) shows that the normalized insertion depth $M_i^{\star} = \frac{M_i - log(i)}{\sqrt{log(i)}}$ has the limiting distribution $\mathcal{N}(0,1)$, the standard normal distribution. Hence, for every $\epsilon' > 0$ and $\tau > 0$, there exists $\rho_{\epsilon',\tau}$ such that for every $\rho > \rho_{\epsilon',\tau}$:

$$\sum_{T:|T|=\rho} \sum_{j \in T} \frac{\mathbb{1}\left(d_T\left(SO, j\right) > \tau\right)}{(\rho - 1)!} > (1 - \epsilon')\, \rho \tag{9}$$

Since the agents are equally likely to meet new entrants, for every $T \in \mathbb{T}$ it must be that $Pr\left(T^n = T : \sigma^n\right) = \frac{Pr(|T^n| = |T| : \sigma^n)}{(|T|-1)!}$ (i.e., all distribution trees of the same size are equally likely). We can write (8) as

$$\sum_{\rho=1}^{k_n - s_n + 2} \sum_{T:|T|=\rho} \sum_{j \in T} \frac{Pr\left(|T^n| = \rho : \sigma^n\right)}{(\rho - 1)!} \mathbb{1}\left(d_T\left(SO, j\right)\right) > (1 - \epsilon) \sum_{\rho=1}^{k_n - s_n + 2} \rho Pr\left(|T^n| = \rho : \sigma^n\right)$$

By (9), the above inequality holds for large values of ρ. By Lemma 6, for any $\rho \in \mathbb{N}$, $\epsilon' > 0$, and sufficiently large value of n, it holds that $Pr\left(|T^n| < \rho | \sigma^n\right) < \epsilon'$. It follows that the above inequality holds for sufficiently large values of n. ■

Observe that when a distributor j such that $d_T\left(SO, j\right) = \tau$ joins the sales force, the SO has to pay commissions of $a_1^{R_n} + ... + a_{\tau-1}^{R_n}$ to the distributors. For every retail sale made by j the SO has to pay commissions of $b_1^{R_n} + ... + b_{\tau}^{R_n}$ to the distributors, and if j purchases the good, then the SO pays $b_1^{R_n} + ... + b_{\tau-1}^{R_n}$ to the distributors.

We are now ready to construct a cost-benefit ratio for the commissions. Let $C\left(a_{\tau}^{R_n}\right) := \sum_{T \in \mathbb{T}} \sum_{j \in T} Pr\left(T^n = T | \sigma^n\right) \mathbb{1}\left(d_T\left(SO, j\right) > \tau\right)$ be the expected cost that is associated with an increase in $a_{\tau}^{R_n}$. The corresponding cost for $b_{\tau}^{R_n}$ is

$$C\left(b_{\tau}^{R_n}\right) := q \sum_{T \in \mathbb{T}} \sum_{j \in T} Pr\left(T^n = T | \sigma^n\right) \left(\mathbb{1}\left(d_T\left(SO, j\right) > \tau\right) + \mathbb{1}\left(d_T\left(SO, j\right) > \tau - 1\right) \sum_{j'=k_n+1}^{n} \frac{1}{j'} \right)$$

where $q \sum_{j'=k_n+1}^{n} \frac{1}{j'}$ is the expected number of sales per distributor after period k_n.

How effective are these commissions? Let us consider the effect of the different commissions on the marginal agent's (i.e., the k_n-th entrant's) willingness to pay for a license. Denote by l_{τ}^n the expected number of agents at the τ-th level of the subtree of G, rooted at the k_n-th entrant when there are n agents. For example, $l_1^n = \sum_{j=k_n+1}^{n} \frac{1}{j}$ and $l_2^n = \sum_{j=k_n+1}^{n-1} \sum_{j'=j+1}^{n} \frac{1}{jj'}$. Observe that $\beta_1^n = \frac{1}{1+l_1^n}$. Since $p = 0$, the k_n-th entrant (according to his analogy-based expectations) expects to have $l_{\tau}^n \left(\beta_1^n\right)^{\tau}$ distributors at the τ-th level of his downline. An increase in $a_{\tau}^{R_n}$ increases the k_n-th entrant's willingness to pay for a license by $W\left(a_{\tau}^{R_n}\right) := l_{\tau}^{n}\left(\beta_1^n\right)^{\tau}$ and an increase in $b_{\tau}^{R_n}$ increases the k_n-th entrant's willingness to pay for a license by $W\left(b_{\tau}^{R_n}\right) := ql_{\tau}^{n}\left(\beta_1^n\right)^{\tau-1}$.

At the optimum, the k_n-th entrant is willing to pay exactly $\phi^{R_n} + c$ for a license. Given σ^n, the k_n-th entrant's willingness to pay can be induced by various schemes. The scheme R_n is the one that minimizes the SO's expected cost of providing this compensation, conditional on σ^n being played. Under R_n, the SO first uses the commissions for which the cost-benefit ratio $\frac{C}{W}$ is lowest and only then turns to the more expensive ones (e.g., if $\frac{C\left(a_{\tau}^{R_n}\right)}{W\left(a_{\tau}^{R_n}\right)} > \frac{C\left(a_{\tau'}^{R_n}\right)}{W\left(a_{\tau'}^{R_n}\right)}$, then $a_{\tau}^{R_n} > 0$ only if $a_{\tau'}^{R_n} = \phi^{R_n}$). Let us use this observation to better understand the structure of R_n.

Lemma 8 *For every n and $\tau \in \{1, ..., n-1\}$ it holds that $l_1^n l_{\tau}^n \geq 2l_{\tau+1}^n$.*

Proof. Define $l_{\tau,t}^n$ to be the expected number of agents in the τ-th level of the subtree of G rooted at the t-th entrant when there are n agents. Note that $l_{\tau,t}^n = 0$ for every $\tau > n - t$. Observe that for any $t < n$ it holds that $\left(l_{1,t}^n\right)^2 = \left(\sum_{j=t+1}^{n} \frac{1}{j}\right)^2 =$

$$\frac{2}{t+1}\left(\frac{1}{t+2} + ...\right) + \frac{2}{t+2}\left(\frac{1}{t+3} + ...\right) + ... + \frac{2}{(n-1)\,n} + \sum_{j=t+1}^{n} \frac{1}{j^2} = 2l_{2,t}^n + \sum_{j=t+1}^{n} \frac{1}{j^2}$$

We prove the lemma by induction on the size of τ. Assume that $l_{1,t}^n l_{\tau-1,t}^n \geq 2l_{\tau,t}^n$ for every $t \leq n$. Let us show that $l_{1,t}^n l_{\tau,t}^n \geq 2l_{\tau+1,t}^n$. We can write this inequality as

$$l_{1,t}^n\left(\frac{l_{\tau-1,t+1}^n}{t+1} + \frac{l_{\tau-1,t+2}^n}{t+2} + ...\right) \geq 2\left(\frac{l_{\tau,t+1}^n}{t+1} + \frac{l_{\tau,t+2}^n}{t+2} + ...\right) \tag{10}$$

Note that $l_{1,t}^n \geq l_{1,t'}^n$ for every $t' \geq t$ and recall that $l_{1,t}^n l_{\tau-1,t}^n \geq 2l_{\tau,t}^n$ for every t. It follows that (10) holds for any t. To complete the proof, observe that $l_{\tau}^n = l_{\tau,k_n}^n$ for every τ. ■

Lemma 9 *Fix an arbitrary integer $\tau^{\star} > 1$. There exists $n\left(\tau^{\star}\right)$ such that $a_{\tau}^{R_n} = 0$ and $b_{\tau}^{R_n} = 0$ for every pair (n, τ) such that $n > n\left(\tau^{\star}\right)$ and $\tau \in \{2, ..., \tau^{\star}\}$.*

Proof. Let us calculate the cost-benefit ratios. By Lemma 8, $l_1^n l_{\tau}^n \geq 2l_{\tau+1}^n$ for every n and $\tau \leq n - k_n$. It follows that $W\left(a_{\tau}^{R_n}\right) \geq 2W\left(a_{\tau+1}^{R_n}\right)$ and $W\left(b_{\tau}^{R_n}\right) \geq 2W\left(b_{\tau+1}^{R_n}\right)$. Note that $qW\left(a_{\tau}^{R_n}\right) = W\left(b_{\tau}^{R_n}\right)\beta_1^n$ for every n and $\tau \leq n - k_n$. By Lemma 7, for every $\tau^{\star} \in \mathbb{N}$, there exists a sufficiently large $n\left(\tau^{\star}\right)$ such that $\frac{C\left(b_{\tau}^{R_n}\right)}{C\left(b_{\tau+1}^{R_n}\right)}$ and $\frac{C\left(a_{\tau}^{R_n}\right)}{C\left(a_{\tau+1}^{R_n}\right)}$ are arbitrarily close to 1, and $\frac{C\left(a_{\tau}^{R_n}\right)}{C\left(b_{\tau}^{R_n}\right)}$ is arbitrarily close to $\frac{1}{q\left(1+l_1^n\right)}$ for every pair $\tau \leq \tau^{\star}$ and $n > n\left(\tau^{\star}\right)$. Hence, for pairs $\tau \leq \tau^{\star}$ and $n > n\left(\tau^{\star}\right)$, it must be that

$$\frac{C\left(a_{\tau}^{R_n}\right)}{W\left(a_{\tau}^{R_n}\right)} < \frac{C\left(a_{\tau+1}^{R_n}\right)}{W\left(a_{\tau+1}^{R_n}\right)}, \frac{C\left(b_{\tau}^{R_n}\right)}{W\left(b_{\tau}^{R_n}\right)} < \frac{C\left(b_{\tau+1}^{R_n}\right)}{W\left(b_{\tau+1}^{R_n}\right)}, \frac{C\left(a_{\tau}^{R_n}\right)}{W\left(a_{\tau}^{R_n}\right)} < \frac{C\left(b_{\tau+1}^{R_n}\right)}{W\left(b_{\tau+1}^{R_n}\right)}, \text{ and } \frac{C\left(b_{\tau}^{R_n}\right)}{W\left(b_{\tau}^{R_n}\right)} < \frac{C\left(a_{\tau+1}^{R_n}\right)}{W\left(a_{\tau+1}^{R_n}\right)}$$

The ratios imply that for every $n > n(\tau^\star)$, if $a_\tau^{R_n} > 0$ or $b_\tau^{R_n} > 0$ for $\tau \in \{2, ..., \tau^\star\}$, then $a_1^{R_n} = \phi^{R_n}$ and $b_1^{R_n} = \eta^{R_n}$. If $a_1^{R_n} = \phi^{R_n}$ and $b_1^{R_n} = \eta^{R_n}$, then the SO's profit cannot exceed his own sales and recruitments. Thus, $\pi(R_n)$ cannot exceed $\sum_{j=1}^{n} \frac{(1+B)}{j}$. For large values of n, $\sum_{j=1}^{n} \frac{(1+B)}{j}$ is less than the lower bound for the SO's expected profit, which we provided in (6). Hence, for every $\tau^\star \in \mathbb{N}$, there exists $n(\tau^\star)$ such that $a_\tau^{R_n} = 0$ and $b_\tau^{R_n} = 0$ for each pair (n, τ) such that $n > n(\tau^\star)$ and $\tau \in \{2, ..., \tau^\star\}$. ■

The next lemma shows that if $\tau^\star$ is sufficiently large, then increasing the commissions $a_{\tau^\star+1}^{R_n}, a_{\tau^\star+2}^{R_n}, ..., a_{n-\tau^\star}^{R_n}$ and $b_{\tau^\star+1}^{R_n}, b_{\tau^\star+2}^{R_n}, ..., b_{n-\tau^\star}^{R_n}$ does not increase the k_n-th entrant's willingness to pay for a license by much.

Lemma 10 *For every $\epsilon > 0$, there exists $\tau^\star \in \mathbb{N}$ and $n(\epsilon, \tau^\star)$ such that $\sum_{\tau > \tau^\star} \frac{l_\tau^n}{\left(1+l_1^n\right)^{\tau-1}} < \epsilon$ for every $n > n(\epsilon, \tau^\star)$.*

Proof. By Lemma 4, for sufficiently large n, it must be that $k_n > \bar{\gamma} n$ and, therefore, $l_1^n < log(1/\bar{\gamma}) + 1$. By Lemma 8, $l_1^n l_{\tau-1}^n \geq 2 l_\tau^n$ for every $\tau \leq n - k_n$. It follows that

$$\sum_{\tau > \tau^\star} \frac{l_\tau^n}{\left(1+l_1^n\right)^{\tau-1}} < 0.5\left(log(1/\bar{\gamma}) + 1\right) \sum_{\tau > \tau^\star} \frac{l_{\tau-1}^n}{\left(1+l_1^n\right)^{\tau-1}}$$

Since $l_1^n l_{\tau-1}^n \geq 2 l_\tau^n$, it must be that $\left(log(1/\bar{\gamma}) + 1\right) \sum_{\tau > \tau^\star} \frac{l_{\tau-1}^n}{\left(1+l_1^n\right)^{\tau-1}} < \epsilon$ for large $\tau^\star$. ■

We are now ready to prove the proposition. Consider large values of $\tau^\star$ and n, and assume that $\phi^{R_n} > 0$. We shall show that the SO can increase his expected profit by means of a scheme that does not charge entry fees and does not pay recruitment commissions. Lemma 8 showed that if n is sufficiently large, then $a_\tau^{R_n} = 0$ and $b_\tau^{R_n} = 0$, for every $\tau \in \{2, ..., \tau^\star\}$. Lemma 9 showed that, if $\tau^\star$ is sufficiently large, then at most $\epsilon\left(\phi^{R_n} + 1\right)$ can be provided by the commissions $\left\{b_\tau^{R_n} : \tau > \tau^\star\right\}$ and $\left\{a_\tau^{R_n} : \tau > \tau^\star\right\}$, where $\epsilon > 0$ can be chosen to be arbitrarily close to 0.

Let us eliminate all of the recruitment commissions (i.e., $a_1^{R_n}$ and $\left\{a_\tau^{R_n} : \tau > \tau^\star\right\}$) and the entry fees. This change reduces the SO's cost as well as his revenue. The change in the SO's expected revenue is ϕ^{R_n} multiplied by the expected number of distributors. By (8), for sufficiently large values of n, the SO's expected revenue is reduced by no more than $\phi^{R_n} \frac{C\left(a_1^{R_n}\right)}{1-\epsilon}$.

Eliminating the recruitment commissions reduces the SO's expected cost by more than $a_1^{R_n} C\left(a_1^{R_n}\right)$, which is the reduction due to the fact we eliminated $a_1^{R_n}$. In addition to this change, let us reduce $b_1^{R_n}$ by $\phi^{R_n} \frac{\epsilon}{1-\epsilon} \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)} + \left(\phi^{R_n} - a_1^{R_n}\right) \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)}$. The total expected reduction in the SO's cost is greater than

$$a_1^{R_n} C\left(a_1^{R_n}\right) + \phi^{R_n} \frac{\epsilon}{1-\epsilon} \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)} C\left(b_1^{R_n}\right) + \left(\phi^{R_n} - a_1^{R_n}\right) \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)} C\left(b_1^{R_n}\right)$$

Since $d_T\left(SO, j\right) > 0$ for every distributor j, it follows that $\frac{C\left(a_1^{R_n}\right)}{W\left(a_1^{R_n}\right)} < \frac{C\left(b_1^{R_n}\right)}{W\left(b_1^{R_n}\right)}$ and, therefore, the above expression is greater than $\phi^{R_n} \frac{C\left(a_1^{R_n}\right)}{1-\epsilon}$.

If the k_n-th entrant is willing to purchase a license after these changes, this will contradict the profit-maximizing property of R_n. Hence, it is left to show that the reduction in the size of the commissions does not reduce the k_n-th entrant's willingness to pay for a license by more than ϕ^{R_n}. First, the elimination of the recruitment commissions reduces his willingness to pay by at most $a_1^{R_n} W\left(a_1^{R_n}\right) + \phi^{R_n}\epsilon$. The reduction in $b_1^{R_n}$ reduces his willingness to pay for a license by at most $\left(\phi^{R_n} - a_1^{R_n}\right) W\left(a_1^{R_n}\right) + \phi^{R_n} \frac{\epsilon}{1-\epsilon} W\left(a_1^{R_n}\right)$. By Lemma 4, $W\left(a_1^{R_n}\right)$ is bounded below 1 (as $l_1^n < log\left(1/\bar{\gamma}\right) + 1$) such that for sufficiently small ϵ the reduction in the k_n-th entrant's willingness to pay for a license is less than ϕ^{R_n}. Thus, we obtain a contradiction to the assumption that R_n is profit maximizing and that $\phi^{R_n} > 0$.

It is left to verify that $b_1^{R_n} \geq \phi^{R_n} \frac{\epsilon}{1-\epsilon} \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)} + \left(\phi^{R_n} - a_1^{R_n}\right) \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)}$ such that the above exercise is viable. If this inequality does not hold, then the k_n-th entrant's willingness to pay for a license under σ^n cannot exceed

$$\phi^{R_n} \frac{\epsilon}{1-\epsilon} \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)} W\left(b_1^{R_n}\right) + \left(\phi^{R_n} - a_1^{R_n}\right) \frac{W\left(a_1^{R_n}\right)}{W\left(b_1^{R_n}\right)} W\left(b_1^{R_n}\right) + \epsilon\phi^{R_n} + \epsilon + a_1^{R_n} W\left(a_1^{R_n}\right)$$

Recall that $\phi^{R_n} \leq B$. Hence, if $\epsilon > 0$ is sufficiently small, the above expression is less than $\phi^{R_n} + c$, which is in contradiction to the optimality of the k_n-th entrant's decision to purchase a license.

In conclusion, for large values of n, the profit-maximizing scheme R_n cannot charge entry fees, for otherwise the SO could increase his expected profit by means of an IC reward scheme that does not charge entry fees. Since R_n is IC and does not charge entry fees, it follows that it does not pay recruitment commissions.

Proof of Proposition 4

Let R be an IC scheme and consider an agent j who purchases a license in period $t \in \mathbb{N}$ of $\Gamma\left(R\right)$. Since R is IC, j's reward cannot exceed $\left(\left|G_j\right| - 1\right) \phi^R - \phi^R$. Since the agents are equally likely to meet each new entrant, in period t agent j believes that, in expectation, $\left|G_j\right| = \sum_{i=1}^{\infty} \frac{\delta_j^i}{t+1} + 1$. Clearly, $\sum_{i=1}^{\infty} \frac{\delta_j^i}{t+1} \leq \sum_{i=1}^{\infty} \frac{\bar{\delta}^i}{t+1}$. The RHS of this equation approaches 0 at the $t = \infty$ limit. Hence, there is a period $t^\star$ from which point onward purchasing a license is strictly suboptimal regardless of j's beliefs about his successors' behavior, even if he holds the most optimistic prior $\bar{\delta}$.

We can use a standard backward induction argument to show that, in an SPE of a game induced by an IC scheme, agents never purchase a license (independently of whether they use the other agents' discount factors or put themselves in the other agents' shoes and use their own discount factor when predicting the other agents' behavior).

Proof of Proposition 5

Let us show that there exists a period $t^\star \in \mathbb{N}$ such that in every game that is induced by an IC scheme, in every $t \geq t^\star$, purchasing a license is strictly suboptimal for the t-th entrant regardless of his beliefs about the other agents' strategies. Consider an agent j who enters a game that is induced by an IC scheme R in period t. His expected number of successors (i.e., members of $G_j - \{j\}$) is $\sum_{i=1}^{\infty} \frac{\delta^i}{t+1}$. Since R is IC, even if every agent who enters the game after period t purchases a license, in expectation, agent j's reward cannot exceed $\sum_{i=1}^{\infty} \frac{\delta^i}{t+1} \left(\phi^R + q\right)$. For large values of t, it holds that $\sum_{i=1}^{\infty} \frac{\delta^i}{t+1} \left(\phi^R + q\right) < c + \phi^R$ for every $\phi^R \geq 0$.

The rest of the proof is identical to the proof of Theorem 4 (replacing n with $t^\star$ in the last paragraph).

Proof of Proposition 6

Let R be an IC one-level scheme, consider an ABEE (σ, β) of $\Gamma(R)$ in which the agents purchase licenses, and denote by $k^\star$ the last period in which an agent purchases a license in this ABEE (if the agents purchase licenses, such a period exists by the argument made in the proof of Proposition 5, which is independent of the agents' beliefs about the other players' behavior).

Since R is a one-level scheme, the distributors' rewards depend only on the first level of their downline. Hence, in an ABEE, every distributor i makes an offer in period t if and only if $\beta\left(M_1^{-i}\right) \Delta_t \leq 1$, where Δ_t is the expected number of the t-th entrant's successors that i will make offers to if (i) he does not sell a license to the t-th entrant, (ii) he makes an offer to every agent whom he interacts with after period t, and (iii) each of the t-th entrant's successors accepts every offer with probability $\beta\left(M_1^{-i}\right)$. Formally, $\Delta_t = \sum_{j=1}^{\infty} l_{j,t} \left(1 - \beta\left(M_1^{-i}\right)\right)^{j-1} p^j$, where $l_{j,t}$ is the expected number of agents in the j-th level of the subtree of G rooted at the t-th entrant, conditional on the game reaching period t. Note that $l_{j,t}$ in weakly decreasing in t, and that, in every ABEE,

$\beta\left(M_1^{-i}\right)=\beta\left(M_1^{-j}\right)$ for each pair of agents $i,j\in I$. Hence, in the ABEE (σ,β), there exists period $\hat{t}\in\mathbb{N}$ such that every distributor makes (respectively, does not make) an offer in each period $t\geq\hat{t}$ (respectively, $t<\hat{t}$).

For every $i\in I$, denote by v_i the expected number of offers that i makes to the members of $\hat{G}_i^{k^\star}=G_i-\cup_{j\in G_i^{k^\star}-\{i\}}G_j$, conditional on holding a license at the end of period $k^\star$. Since the agents are equally likely to meet new entrants, and since every distributor makes an offer to every agent whom he interacts with in each period $t\geq\hat{t}$ and does not make any offers prior to period $\hat{t}$, it follows that $v_i=v_j=v$ for every pair of agents $i,j\in I$. Hence, every offer that is accepted at time $t\leq k^\star$ results in a distributor l who makes, in expectation, v offers to the members of $\hat{G}_l^{k^\star}$ after period $k^\star$ (conditional on the game reaching period $k^\star$, which occurs with probability $\delta^{k^\star-t}$). Thus, in expectation, for every offer that is accepted at $t\leq k^\star$, there are at least $\delta^{k^\star}v$ offers that are rejected after period $k^\star$. As a result, $\beta\left(M_1^{-i}\right)\leq\frac{1}{1+\delta^{k^\star}v}$ for each $i\in I$.

As we showed in Theorem 1, every agent who purchases a license in period $k^\star$ underestimates the number of offers that he will make after period $k^\star$ (he believes that he will make $\tilde{v}\leq v$ offers since he underestimates the likelihood that future entrants will reject his offers). Hence, the $k^\star$-th entrant, j, believes that, in expectation, he will sell no more than $\beta\left(M_1^{-j}\right)v$ licenses. Note that v cannot exceed the expected number of agents in the subtree of G rooted at the $k^\star$-th entrant (excluding the root), $\frac{1}{k^\star+1}\sum_{t=1}^{\infty}\delta^t$. It follows that $\beta\left(M_1^{-j}\right)v\leq\frac{v}{1+\delta^{k^\star}v}\leq\frac{1}{(1-\delta)(k^\star+1)+\delta^{k^\star}}<1$. This completes the proof as, if R is IC, then the $k^\star$-th entrant cannot analogy-based expect a payoff greater than $\frac{1}{(1-\delta)(k^\star+1)+\delta^{k^\star}}\phi^R-\phi^R<0$ if he purchases a license. This is in contradiction to the assumption that an agent purchases a license in period $k^\star$ in the ABEE (σ,β).

Proof of Proposition 7

Set $\alpha=1$, $p=0$, and consider the set of two-level IC reward schemes $\mathcal{M}$ and the profile of strategies σ that were presented in the proof of Proposition 2. Let us verify that if $\delta<1$ is sufficiently large, then σ is part of an ABEE of $\Gamma(R)$ for some $R\in\mathcal{M}$.

First, consider an arbitrary $R\in\mathcal{M}$ and $\delta<1$. Since $p=0$, every distributor finds it optimal to make an offer to every agent whom he meets. The SO is indifferent between making an offer and not doing so in each period $t>1$, and finds it optimal to make an offer in period 1. It is left to verify that the agents find it optimal to purchase a license in period $t=1$ and not to purchase one in each period $t>1$.

Let us consider the agents' analogy-based expectations. First, it is easy to see that $\beta\left(M_2^{-i}\right)=1$ is consistent with σ for each $i\in I$. Second, in expectation, the first entrant makes $\sum_{i=1}^{\infty}\frac{\delta^i}{1+i}$ offers, which are all rejected (recall that he makes an offer to every agent whom he meets directly and, since $p = 0$, only to these agents). Thus, $\beta\left(M_1^{-i}\right) = \frac{1}{1+\sum_{i=1}^{\infty}\frac{\delta^i}{1+i}}$ for each $i \in I$.

Since $p = 0$, the first entrant correctly expects to make $\sum_{i=1}^{\infty}\frac{\delta^i}{1+i}$ offers. He analogy-based expects to sell $\beta\left(M_1^{-i}\right)\sum_{i=1}^{\infty}\frac{\delta^i}{1+i} = \frac{log\left(\frac{1}{1-\delta}\right)-\delta}{log\left(\frac{1}{1-\delta}\right)}$ licenses if he purchases one. Observe that this expression approaches 1 at the $\delta = 1$ limit. The first entrant believes (according to his analogy-based expectations) that, in expectation, he will have

$$\frac{\sum_{j=1}^{\infty}\sum_{j'=j+1}^{\infty}\delta^{j'}\frac{1}{(1+j)(1+j')}}{\left(\frac{1}{\delta}log\left(\frac{1}{1-\delta}\right)\right)^2} = \frac{\sum_{j=2}^{\infty}\delta^j\frac{H_j-1}{(1+j)}}{\left(\frac{1}{\delta}log\left(\frac{1}{1-\delta}\right)\right)^2} = \frac{\frac{1}{2\delta}log^2\left(\frac{1}{1-\delta}\right) - \frac{1}{\delta}log\left(\frac{1}{1-\delta}\right) + 1}{\left(\frac{1}{\delta}log\left(\frac{1}{1-\delta}\right)\right)^2} \tag{11}$$

distributors at the second level of his downline, where H_j denotes the j-th harmonic number. It is easy to see that the RHS of (11) approaches $\frac{1}{2}$ as δ goes to 1. The rest of the proof is identical to the proof of Proposition 2, from which Theorems 2.1 and 3 follow.

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