# Multilevel Marketing: Pyramid-Shaped Schemes or Exploitative Scams?*

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**Abstract**

Motivated by the growing discussion on the resemblance of multilevel marketing schemes to pyramid scams, we compare the two phenomena based on their underlying compensation structures. We show that a company can design a pyramid scam to exploit a network of boundedly rational agents. We find that maximizing the company's profit from a pyramid scam requires charging the participants a license fee and paying them a recruitment commission for each of the people that they recruit and that their recruits recruit. We characterize the schemes that maximize a company's profit when it faces fully rational agents, and establish that the company never finds it profitable to charge them a license fee or pay them recruitment commissions.

What delineates pyramid scams from legitimate multilevel marketing enterprises? Recent growth[1] in the multilevel marketing (MLM) industry—which over the past five years has engaged over 20 million[2] Americans—has raised the urgency of this question for consumer protection agencies. MLM companies such as Avon, Amway, Herbalife, and Tupperware use independent representatives to sell their products to friends

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[1]Membership in the MLM industry is substantial and growing. For example, the global force of independent distributors reached nearly 117 million in 2017 (World Federation of Direct Selling Associations, 2018).

[2]According to the Direct Selling Association's (DSA) annual report (DSA, 2016).

and acquaintances. They all promote the opportunity of starting one's own business and making extra income; however, some (e.g., Bort, 2016) view these companies as pyramid scams whose main purpose is to take advantage of vulnerable individuals.

The MLM industry's questionable legitimacy received considerable media attention following a recent FTC investigation against Herbalife (FTC, 2016a). Identifying whether a particular company is a legitimate one, or whether it is an exploitative pyramid scam that promotes useless products and services in order to disguise itself as a legitimate firm, can be a daunting task. One obstacle is that MLM companies typically sell products whose quality is difficult to assess, such as vitamins and nutritional supplements. The common wisdom among practitioners is that a company is legitimate if the distributors are encouraged to sell the product, and it is an illegal pyramid scam if it prioritizes recruitment over selling (FTC, 2016b; SEC, 2013). However, it is extremely difficult to determine the company's true "selling point" and, in practice, it is challenging to distinguish between sales to members and sales to the general public.

The objective of this paper is to draw the boundary between the two phenomena on the basis of their underlying compensation schemes. Our premise is that the potential distributors are strategic, and that the MLM company chooses a compensation scheme while taking these prospective distributors' incentives into account. To understand the structure of potential reward schemes, consider the following example.

**Example 1** *The reward scheme $R$ pays every distributor a commission of $b_1$ for every sale he makes and a commission of $a_1$ for every agent he recruits to the sales force. The reward scheme $R'$ pays every distributor a commission of $b'_1$ for every sale he makes and a commission of $b'_2$ for every sale one of his recruits makes. It also pays every distributor a commission of $a'_1$ for every agent he recruits and a commission of $a'_2$ for every agent one of his recruits recruits. Both schemes charge a license fee[3] of $\phi \geq 0$ from every distributor. We refer to $a_1, a'_1$, and $a'_2$ as recruitment commissions, and to reward schemes such as $R$ (respectively, $R'$) as 1-level (respectively, multilevel) schemes as they compensate the distributors based on the first level (respectively, multiple levels) of their downline.*

Both $R$ and $R'$ compensate the distributors for recruiting others to work for the company. In practice, however, over 90% of the network marketing industry uses multilevel schemes such as $R'$ (DSA, 2014). Moreover, although there is no obvious reason why

[3] In practice, the license fee is often presented as a training cost or a requirement to purchase initial stock.

1-level schemes cannot be used for the purpose of sustaining a pyramid scam, various companies that were deemed[4] pyramid scams used multilevel reward schemes. What can explain these stylized facts? Can a "legitimate" company benefit from charging license fees, paying recruitment commissions, or offering multiple routes through which individuals can join the sales force? Does the answer depend on whether the company promotes genuine goods or merely the opportunity to recruit others to the sales force?

To address the above questions, we develop a model in which a scheme organizer (SO) sells a good to a network of agents that is formed randomly and sequentially, where the agents buy only from people to whom they are directly connected. To reach new pools of customers (agents to whom the SO is not directly connected), the SO uses a reward scheme to incentivize agents to sell the good and recruit (sell distribution licenses to) new distributors. A key feature of the model is that each agent's likelihood of meeting new potential buyers and distributors decreases as time progresses, which makes it unattractive to join the sales force late in the game.

To capture the idea that the main product that is being traded in a pyramid scam is the right to recruit other agents to the pyramid, assume for a moment that the good has no intrinsic value such that the only "products" being traded in the model are distribution licenses. If there exists a reward scheme such that the SO makes a strictly positive profit in its induced game, then we have a *pyramid scam*. Classic no-trade theorems rule out such scams for rational agents in our model, reflecting the fact that such agents cannot be fooled.[5] Hence, to better understand such scams and their underlying reward schemes, we depart from the rational expectations paradigm.

In the main part of the paper, we use the analogy-based expectation equilibrium (ABEE) framework (Jehiel, 2005) to relax the requirement that each agent has a perfect perception of the other agents' behavior. Under the behavioral model, agents neglect the fact that other agents' strategies are time-contingent. As a result, they underestimate the extent to which recruiting new members becomes more difficult over time. Despite this mistake, each agent's beliefs are statistically correct and can be interpreted as resulting from the use of a simplified model of the other agents' behavior.

We establish that if there are sufficiently many agents, then the SO can sustain a pyramid scam; that is, there are reward schemes that enable the SO to make a strictly

[4]See, e.g., *FTC v. Fortune Hi-Tech Marketing Inc.* (2013) for a pyramid scam that used a multilevel reward scheme to recruit over 100,000 distributors in the United States and Canada.

[5]While under the classic rational expectations paradigm agents cannot be scammed, in practice, we observe countless pyramid scams (see, e.g., Keep and Vander Nat, 2014, and the references therein).

positive expected profit in an ABEE even if the good has no intrinsic value. After establishing the existence of such schemes, we study their structure and show that they (i) charge a license fee and (ii) pay for at least two levels of downline recruits. In other words, there exists no 1-level scheme such that the SO makes a strictly positive expected payoff in its induced game when the only products being traded are distribution licenses.

While the intuition for the existence result is similar to the intuition in previous environments in which ABEE has been used to explain speculation—e.g., the centipede game (Jehiel, 2005) or the capped bubble game (Moinas and Pouget, 2013)—the intuition for the impossibility result is new. As in previously studied environments, ABEEs have a threshold feature: agents buy licenses up to some time $t$ and then stop. We show that, conditional on buying a license at time $t$, an agent whose beliefs are statistically correct (as in an ABEE) cannot expect to sell more than one license. Thus, no commission on the agent's own sales (of licenses) would cover the license fee and make it beneficial for him to participate. Note that, in addition to overestimating the number of direct recruits, our agent also overestimates the number of agents his recruits will recruit. When the SO uses a multilevel scheme, these mistakes accumulate, and so the agent may find it worthwhile to purchase a license.

We further investigate the features of cognitive biases that help sustain fraudulent scams by incorporating into our framework a behavioral model in which agents' beliefs are not necessarily statistically correct. Specifically, in Section 6, we adapt Brunnermeier and Parker's (2005) model of self-deception to our framework. We show that even when agents deceive themselves into holding extremely overoptimistic beliefs about their location in the game tree (e.g., the belief that they are likely to be "at the top of the pyramid") and, as a result, it is possible for the SO to sustain a pyramid scam by means of a 1-level scheme, the scam that maximizes the SO's expected profit is built on a multilevel scheme.

To obtain a better understanding of MLM, we investigate a setting in which the good is intrinsically valued. We solve for the SO's optimal scheme under two behavioral assumptions. First, if the agents are fully rational, then the optimal scheme does not charge license fees, nor does it pay recruitment commissions. Second, when agents are analogy-based reasoners, then the properties of the optimal scheme depend on the demand for the good. When the demand is sufficiently high, the optimal scheme looks just like when agents are fully rational. However, when the demand is sufficiently low, the optimal scheme charges a license fee and pays for at least two levels of downline recruits. Thus, the tools that pyramid scams are based on—recruitment commissions and license fees—disappear when the demand for the good is high and emerge again when the demand is low.

The main contribution of this paper is in explaining why certain MLM compensation structures, and in particular MLM compensation structures that emphasize license fees and recruitment commissions, appear to be correlated with scams. The explanation is that license fees and multiple levels of recruitment commissions increase scam profits when agents are boundedly rational and, in some instances, make scams viable rather than not, whereas, in a baseline case with rational agents, these tools are not useful for legitimate businesses. We study the implications of banning license fees and recruitment commissions and find that such a regulation may reduce the profit of an SO who faces analogy-based reasoners. However, we show that there is a limit to this negative effect. Even under such a regulation, the SO can still obtain an expected profit that is at least as high as the *fundamental value* of the operation, namely, the expected profit that an unregulated SO could obtain when facing fully rational agents. Thus, such a regulation could eliminate distributor losses and would never lead a business that can survive with rational participants to shut down.

*Related literature*

Our paper relates to the strand of the behavioral industrial organization literature that studies market settings and contractual features that enable firms to exploit agents who are subject to different biases.[6] Spiegler (2011) offers a textbook treatment of such models and Heidhues and Kőszegi (2018) provide a comprehensive review of this strand of the literature. In Eliaz and Spiegler (2006, 2008), a principal interacts with agents who differ in their ability to predict their future tastes. In Gabaix and Laibson (2006), firms may hide information about add-on prices from unaware consumers. Grubb (2009) studies contracting when agents are overconfident about the accuracy of their forecasts of their own future demand. Crawford et al. (2009) and Jehiel (2011) study manipulative auction design. Heidhues and Kőszegi (2010) study exploitative credit contracts when consumers are time-inconsistent. Grubb (2015) illustrates how various contractual features can be used to exploit overconfidence (e.g., automatic renewal).

[6]A related strand of the literature studies the vulnerability of nonstandard preferences or of nonstandard decision-making procedures to exploitative transactions ("Dutch Books"). See, e.g., Laibson and Yariv (2007) and Rubinstein and Spiegler (2008).

Although this literature is not insubstantial, the various contractual features inherent in MLM are not fully understood.

In the context of MLM, an independent paper by Stivers et al. (2019) studies important aspects in the relation between MLM schemes and pyramid scams. They assume that agents are overoptimistic about the possibility of recruiting new members. As a result of this overoptimism, the agents purchase distributorships regardless of whether they believe that retailing the good is profitable, which results in a pyramid scam. There are two main differences between their model and ours. First, Stivers et al. (2019) take a reduced-form approach to how agents form their beliefs while in the present paper agents' beliefs emerge endogenously in equilibrium. This endogeneity constrains our agents' ability to err and allows us to investigate the nature and extent of biases that are needed to support a pyramid scam. Second, Stivers et al. (2019) restrict attention to one-level reward schemes while the present paper allows for a richer set of schemes. This richness allows us to characterize optimal "legitimate" MLM structures and optimal pyramid scam structures. In particular, we find that even in instances in which one-level pyramid scams are viable, they do not maximize the SO's profit.

We use the analogy-based expectation equilibrium (Jehiel, 2005) as our main behavioral framework. A closely related concept, the partially cursed equilibrium, was developed by Eyster and Rabin (2005) for Bayesian games. In a partially cursed equilibrium, each agent fails to realize the extent to which the other agents' actions depend on their private information. Similar ideas have been applied in various contexts and, in particular, in the context of behavioral industrial organization. For example, Piccione and Rubinstein (2003) study intertemporal pricing when consumers reason in terms of a coarse representation of the correct equilibrium price distribution.

The pure pyramid scams in our model resemble speculative bubbles (note that bubbles do not include the design and recruiting aspects of a pyramid). Shiller (2015) describes such bubbles as naturally occurring Ponzi processes.[7] Bianchi and Jehiel (2010) and Moinas and Pouget (2013) show that the analogy-based expectation equilibrium logic can sustain a bubble, and Jehiel (2005) shows that it can sustain cooperation in the finite-horizon centipede game, which can be interpreted as a speculative bubble.

Abreu and Brunnermeier (2003) and Moinas and Pouget (2013) study models in

[7] Although Ponzi schemes and pyramid schemes are related and these terms are often used synonymously, they are different in several important aspects. In particular, Ponzi scheme participants are not required to recruit new members in order to make a profit. Moreover, they sometimes believe that an ordinary investment underlies the operation. For example, the participants in Wincapita, a Finnish Ponzi scheme, did not know the true nature of the scheme until it collapsed (Rantala, 2019).

which investors become aware of a finite bubble sequentially and face uncertainty about the time at which the bubble started. In both models, a bubble can be sustained in equilibrium under common knowledge of rationality. Unlike in these models, in the present paper classic no-trade arguments (Tirole, 1982) hold and so uncertainty about the time at which the operation started cannot lead to participation in a pure pyramid scam without deviations from the rational expectations paradigm. The reason that no-trade arguments do not hold in the previous models is either that the induced trading game is not a negative-sum game (in Abreu and Brunnermeier's model) or that traders can suffer a (potentially) infinite loss (in Moinas and Pouget's model). In our game, on average, pyramid scam participants incur losses, and their potential loss is bounded.

MLM schemes have received considerable attention outside of the economics literature. A strand of the computer science literature (see, e.g., Emek et al., 2011; Babaioff et al., 2012) focuses on MLM schemes' robustness to Sybil attacks. The marketing literature has addressed ethical issues in MLM and the resemblance of such schemes to pyramid scams. The common view in that literature is that a company is a pyramid scam if the participants' compensation is based primarily on recruitment rather than sales to end users (see, e.g., Koehn, 2001; Keep and Vander Nat, 2002).

The paper proceeds as follows. The model is presented in Section 1. Sections 2 and 3 provide benchmark results about the social optimum and the SO's optimal scheme when agents are fully rational. Section 4 investigates pure pyramid scams and Section 5 studies MLM using the ABEE framework, Section 6 studies an alternative behavioral explanation, and Section 7 concludes.

# 1 The Model

There is a scheme organizer (SO) who produces a good free of cost and with no capacity constraints, and a set of agents $I = \{1, ..., n\}$. Each agent $i \in I$ has a unit demand and a willingness to pay $\omega_i \in \{0, 1\}$ that is drawn by nature when $i$ enters the game. For each $i \in I$, denote $q := Pr(\omega_i = 1)$.

In each period $t = 1, 2, ..., n$, one agent enters the game. We refer to the $t$-th entrant as agent $i_t$. Upon entering the game, agent $i_t$ meets one player $j \in \{SO, i_1, ..., i_{t-1}\}$ chosen uniformly at random by nature.[8] For example, agent $i_2$ meets either the SO or agent $i_1$, each with probability 0.5. In period 1, the SO can offer agent $i_1$ the opportunity to purchase a distribution license. Conditional on receiving an offer, $i_1$

[8] We assume that nature makes her choice at the beginning of period $t$.

can accept or reject it. Let $D_1 = \{SO, i_1\}$ if $i_1$ accepts an offer and $D_1 = \{SO\}$ otherwise. In each period $t > 1$, if agent $i_t$ meets a player $j \in D_{t-1}$, then $j$ can offer him the opportunity to purchase a distribution license. If he receives an offer, $i_t$ can accept or reject it. Let $D_t = D_{t-1} \cup \{i_t\}$ if $i_t$ accepts an offer and $D_t = D_{t-1}$ otherwise.

Let $G$ denote the directed tree, rooted at the SO, that is induced by the above process, where each node represents an agent and each meeting is represented by an edge that points away from the root. We use $d(i, j)$ to denote the length of the directed path from $i \in I \cup \{SO\}$ to $j \in I$ if such a path exists.

To illustrate the game tree, suppose that, at the end of period 9, $G$ is as presented in Figure 1 and $D_9 = \{SO, i_1, i_2, i_3\}$. Recall that agent $i_{10}$ is equally likely to meet each $j \in \{SO, i_1, ..., i_9\}$. If $i_{10}$ meets $j \in D_9$, then $j$ decides whether or not to make $i_{10}$ an offer. If $i_{10}$ meets an agent $j \notin D_9$, then agent $i_{10}$ does not receive an offer.

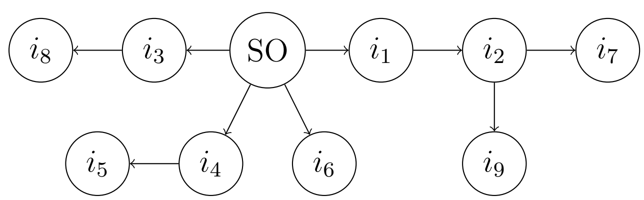


Figure 1: A snapshot of $G$ at the end of period 9.

Agents in the model make two types of strategic choices: whether to accept an offer upon entering the game, and, conditional on purchasing a license, whether to make offers when they meet agents later in the game. The choice of whether to purchase the good for personal consumption is modeled in a nonstrategic manner, as will be described in the next paragraph. In each period $t$, every player $i \in \{SO, i_1, ..., i_t\}$ knows the period $t$, his immediate predecessor in $G$, the realization of the subtree of $G$ rooted at $i$ up to period $t$, and, for every agent $j$ in that subtree, whether $j$ accepted an offer and whether $j$ bought the good. Denote by $H_i$ the set of information sets in which player $i \in I \cup \{SO\}$ moves. Player $i$'s strategy is a mapping $\sigma_i : H_i \to \{0, 1\}$, where in each $h \in H_i$, $i$ chooses whether or not to make an offer, or else chooses whether or not to accept one. We use $\sigma = (\sigma_i)_{i \in I \cup \{SO\}}$ to denote a profile of strategies.

Distributors are paid according to a reward scheme that consists of:

- A license fee $\phi \geq 0$.
- Recruitment commissions: $a_1, a_2, a_3, ... \geq 0$.
- Sales commissions: $b_1, b_2, b_3, ... \geq 0$.

- A price $\eta > 0$ for which each unit of the good is sold.

When agent $i_t$ accepts an offer, he pays $\phi$ to the SO and the SO pays a commission of $a_{d(l,i_t)}$ to every distributor $l$ on the directed path from the SO to $i_t$. We assume that, for each $t \in \{1, ..., n\}$, agent $i_t$ upon entering the game purchases a unit of the good for personal consumption if and only if both (i) $\omega_{i_t} \geq \eta$ and (ii) $i_t$'s immediate predecessor in $G$ is a member of $D_{t-1}$ (i.e., a distributor or the SO). If agent $i_t$ purchases the good, then he obtains an additional utility of $\omega_{i_t}$, pays $\eta$ to the SO, and the SO pays a commission of $b_{d(l,i_t)}$ to every distributor $l$ on the directed path from the SO to $i_t$. In addition to the commissions, when an agent accepts an offer, he incurs a cost of $c > 0$, which reflects the cost of learning how to market the good, and the person who recruited him incurs a cost of $\hat{c} > 0$, which reflects the cost of training the new distributor.[9]

We denote by $\Gamma(R)$ the game that is induced by the reward scheme $R$. Agents maximize their expected payoff in $\Gamma(R)$ given the reward scheme and their beliefs about other players' behavior. The highest expected payoff the SO obtains across all equilibria of $\Gamma(R)$ is denoted by $\pi(R)$.

The SO faces the risk that the distributors will create fictitious recruits in order to become eligible for additional commissions.[10] Motivated by this risk, we shall focus on schemes where $a_\tau \leq \phi$ and $b_\tau \leq \eta$ for every $\tau \geq 1$, and refer to such schemes as *incentive-compatible* (IC) schemes. This constraint implies that for a distributor, the cost of creating a fictitious new tree of sales and recruits is greater than the direct benefit of doing so (i.e., the transfers from the SO to the root). The incentive-compatibility constraint rests on the assumption that the SO can verify[11] the identity of any distributor who wishes to receive commissions and, therefore, even if a distributor were to create a fictitious recruit, he would not be able to collect the commissions that the fictitious recruit is eligible to receive.

*Discussion: Meeting process*

We borrow the meeting process from the applied statistics literature, where it is referred to as the *uniform random recursive tree model* (for a textbook treatment, see Drmota,

[9] It is possible to incorporate moral hazard into the model by adding a choice of whether to train the new recruit without changing the key results of the paper.

[10] In the computer science literature, manipulations in this spirit are often referred to as local false-name manipulations or local splits (see, e.g., Emek et al., 2011; Babaioff et al., 2012).

[11] We shall discuss the verifiability assumption in detail in the concluding section.

2009).[12] This process rests on the assumptions that there is a deterministic date at which the game ends and that each agent is directly connected to only one agent upstream (i.e., agents have market power in their social neighborhood). Our main results do not depend on these assumptions or on the specific process. Nonetheless, we use this process since it allows us to convey the main messages while keeping the exposition simple. In a previous version of this paper (Antler, 2021, pp. 43-50), it is shown that the main insights hold when there is uncertainty about the length of the game as well as when agents can be connected to multiple agents upstream.

# 2 The Social Optimum

From a social perspective, the cost of turning agent $i_t$ into a distributor is $c + \hat{c}$: $c$ is incurred by $i_t$ when learning how to market the good, and $\hat{c}$ is incurred by his recruiter when training $i_t$. Recall that agents consume the good only if they meet a distributor upon entering the game. In expectation, agent $i_t$ meets

$$v_t := E\Big[|\{j \in I : d(i_t, j) = 1\}|\Big] = \frac{1}{t+1} + ... + \frac{1}{n} \tag{1}$$

new entrants in periods $t+1, ..., n$ (for completeness, let $v_n := 0$). Thus, if agent $i_t$ becomes a distributor he can provide the good to $v_t$ agents who would not consume it otherwise, where $q$ of them are willing to pay 1 for the good. Hence, the direct social benefit from turning agent $i_t$ into a distributor is $qv_t$. This cost-benefit analysis leads to the following definition.

**Definition 1** *A reward scheme $R$ is said to be socially optimal if there exists an equilibrium of $\Gamma(R)$ in which, in every period $t \in \{1, ..., n\}$, if $qv_t > c + \hat{c}$, then the entrant purchases a license and, if $qv_t < c + \hat{c}$, then he does not purchase a license.*

Note that Definition 1 takes into account only direct sales. If $qv_t > c + \hat{c}$, then it is socially desirable that agent $i_t$ will purchase a license even if we do not take into account downline sales. Since $v_t$ is monotonically decreasing in $t$, if $qv_t < c + \hat{c}$, then from a social perspective it is best if agent $i_t$ does not recruit anyone. Thus, when looking for a socially optimal scheme we can ignore downline sales and recruitment.

[12]Gastwirth (1977) and Bhattacharya and Gastwirth (1984) used this model to examine two real-world pyramid scams and to demonstrate that only a small fraction of the participants can cover the license fees. In none of these papers, however, is there strategic interaction.

# 3 Fully Rational Agents

Our objective in this section is to understand the properties of reward schemes that maximize the SO's expected payoff when he cannot scam the agents. To this end, we assume that agents are fully rational and, hence, not vulnerable to deceptive practices. In this setting, the reward scheme incentivizes agents to sell the good and recruit others to the sales force, which allows the SO to reach potential customers to whom he is not directly connected who would not buy the good otherwise.

To capture that the agents are fully rational, we use perfect Bayesian equilibrium (PBE) to solve the model. A scheme $R$ is said to be profit-maximizing if there exists no IC scheme $R'$ such that $\pi^{PBE}(R') > \pi^{PBE}(R)$. Throughout the analysis, we assume that an agent who is indifferent whether to accept an offer or not accepts it, and that a distributor who is indifferent whether to make an offer or not makes it.[13] To avoid trivial cases in which the reward scheme is essentially irrelevant, we shall assume in this section that $n$ is sufficiently large such that in every PBE of a profit-maximizing scheme at least two agents purchase licenses. It can be shown that this requirement is satisfied when $qv_4 > c + \hat{c}$, where $v_4$ is given in (1) for $t = 4$.

Consider the agents' perspective. Since the likelihood of meeting new entrants goes down over time, agents who enter the game in its early stages meet more people, and so have more opportunities to sell the good and recruit new members. As the cost of purchasing a license is fixed, early entrants find it more beneficial to purchase a license than later entrants. Moreover, since the cost of training a new recruit is fixed, distributors find it more beneficial to recruit a new entrant in the early stages of the game rather than in its later stages (we say that a distributor recruits an agent if he makes an offer to the latter and the latter accepts). The next lemma formalizes this argument and shows that the equilibrium of the game has a threshold structure.[14]

**Lemma 1** *Consider a reward scheme $R$ in which $\eta \leq 1$. In a PBE of $\Gamma(R)$:*

1. *If agent $i_t$ receives an offer, he accepts it if and only if $v_t q b_1 \geq c + \phi$.*

2. *A distributor who meets agent $i_t$ recruits him if and only if $v_t q b_1 \geq c + \phi$ and $v_t q b_2 \geq \hat{c} - a_1$.*

[13] Our results are not sensitive to these assumptions since agents break their indifference in favor of accepting and making offers in the PBE that maximizes the SO's expected payoff.

[14] The lemma assumes that $\eta \leq 1$ as, otherwise, agents would not purchase the good and $\pi^{PBE}(R) = 0$.

Lemma 1 establishes that agents' PBE behavior depends only on $\phi$, $b_1$, $a_1$, and $b_2$. Hence, schemes that pay higher-order commissions can only increase the SO's cost and reduce his profit. Therefore, in the remainder of this section, we shall focus on schemes in which $a_2 = a_3 = ... = 0$ and $b_3 = b_4 = ... = 0$.

Having considered the agents' perspective, we now turn to study the scheme that maximizes the SO's expected payoff.

**Theorem 1** *If $R$ is IC and profit-maximizing, then $\phi = 0 = a_1$.*

To understand why charging a license fee is detrimental to the SO, consider the last agent who is supposed to buy a license in a PBE and denote him by $k$. At the optimum, agent $k$ is indifferent whether to buy a license or not as, otherwise, the SO could increase his profit by charging a higher license fee without affecting the agents' equilibrium behavior. Suppose that the SO eliminates the license fee and, at the same time, lowers the commission $b_1$ such that agent $k$ remains indifferent. This exercise has no effect on the expected net transfers from the SO to agent $k$. However, it reduces the expected net transfers the SO makes to agents who buy a license prior to agent $k$. The reason is that the reduction in $b_1$ has a stronger effect on agents who make more sales while the reduction in $\phi$ is independent of sales. In other words, scaling down $\phi$ and $b_1$ reduces the rents agents obtain by purchasing a license early in the game. Thus, by eliminating the fee and reducing $b_1$ the SO increases his expected payoff.

Observe that IC schemes that do not charge a license fee cannot pay directly for recruitment; that is, $\phi = 0$ implies that $a_1 = 0$. Thus, by refraining from charging a fee the SO loses some flexibility in choosing how to incentivize distributors to recruit new members: he must use $b_2$ rather than $a_1$ or a combination of the two commissions. As we show in the proof, this effect is of second order such that the SO is better off using a scheme that does not charge a license fee and does not pay a recruitment commission.

We can conclude that to maximize his expected payoff, the SO pays $b_1$ for direct sales and $b_2$ for indirect sales. As Lemma 1 shows, the agents' behavior is characterized by two thresholds, $k_1$ and $k_2 \leq k_1$, such that every agent who enters the game up to period $k_1$ accepts every offer he receives, distributors recruit every agent they meet up to period $k_2$, and the SO is the only one making offers in periods $k_2 + 1, ..., k_1$. In a profit-maximizing scheme, agent $i_{k_1}$ must be indifferent whether to accept an offer or not and distributors must be indifferent whether to recruit agent $i_{k_2}$ or not. Hence, by Lemma 1, $b_1 = \frac{c}{qv_{k_1}}$ and $b_2 = \frac{\hat{c}}{qv_{k_2}}$ in a profit-maximizing scheme.

Lemma 1 can also be used to find socially optimal schemes. By Definition 1, if $m = max\{t|qv_t \geq c+\hat{c}\}$, then every scheme where the first $m$ entrants (and only them) purchase a license in a PBE of its induced game is socially optimal. By Lemma 1, any scheme that pays $b_1 = \frac{c+\phi}{qv_m}$ and $b_2 = \frac{\hat{c}-a_1}{qv_m}$ and charges $\eta \leq 1$ incentivizes this behavior. In particular, there exists a socially optimal scheme $\hat{R}$ that pays $\hat{b}_1 = \frac{c}{qv_m}, \hat{b}_2 = \frac{\hat{c}}{qv_m}$, and $\hat{a}_1 = 0$, and charges $\hat{\eta} = 1$ and $\hat{\phi} = 0$. Since $qv_m \geq c + \hat{c}$, $\hat{R}$ is IC. The next corollary summarizes this finding.

**Corollary 1** *There exists an IC socially optimal scheme that does not charge a license fee and does not pay recruitment commissions.*

In general, profit-maximizing schemes are not socially optimal: they pay lower commissions and incentivize fewer agents to become distributors than is socially optimal. Hence, profit-maximizing schemes result in fewer agents buying the good than is socially optimal. To obtain intuition, note that, from the SO's perspective, increasing the number of distributors from $t-1$ to $t$ entails paying higher commissions that make it worthwhile for $i_t$ to buy a license. Since all agents are paid according to the same scheme, by increasing the commissions the SO essentially forgoes some of the profit from the first $t-1$ distributors' sales. The social point of view ignores this effect and, therefore, socially optimal schemes induce more distributors relative to profit-maximizing schemes.

The results in this section show that when agents are fully rational, the SO does not benefit from using recruitment commissions, charging license fees, or paying for more than two levels of sales. Moreover, these components of the reward scheme are unnecessary in order to maximize social welfare. In the following sections, we study settings in which agents are boundedly rational, and investigate in which situations these components of the reward scheme can be useful both from the SO's perspective and from the social perspective.

# 4 Pure Pyramid Scams ($q = 0$)

To capture the idea that the main "good" being traded in a pyramid scam is the right to recruit others to the pyramid, we set $q = 0$, in which case it is commonly known that the only goods that are being traded in the model are distribution licenses.[15]

[15] As will become clear later, the main insights we obtain in this section remain valid when $q > 0$ as long as $q$ is small with respect to $c$ and $\hat{c}$.

Intuitively, such a market should not exist as trade in distribution licenses does not add any value. Indeed, at the social optimum, no agent purchases a license in this case.

We say that the SO sustains a *pure pyramid scam* when $q = 0$ and there exists a scheme $R$ such that the SO makes a strictly positive expected profit in an equilibrium of its induced game. Proposition 1 establishes that the SO cannot sustain a pure pyramid scam when all of the agents are fully rational.

**Proposition 1** *Let $q = 0$. There exists no IC scheme $R$ such that $\pi^{PBE}(R) > 0$.*

When $q = 0$, reward schemes induce negative-sum transfers between the agents and the SO. Proposition 1 then follows from classic no-trade arguments (Tirole, 1982).

Our main objective is to understand the forces and compensation plans that enable pyramid scams to operate. As Proposition 1 shows, we cannot do so by means of the classic rational expectations model and we shall therefore depart from this model.

## 4.1 The baseline behavioral model

Jehiel (2005) suggests an elegant framework that incorporates partial sophistication into extensive-form games. In this framework, different contingencies are bundled into analogy classes and the agents are only required to hold correct beliefs about the other agents' *average behavior* in every analogy class.

Our agents have this type of correct, yet coarse, perception of the other agents' behavior. They understand the frequencies at which the other agents accept and make offers. However, they do not understand the time-contingent nature of the other agents' behavior. In simple words, agents do not base their expectations that offers will be accepted on the time at which they are made. Instead, they pool all offers made at any point in time and consider the average rate of offer acceptances. Thus, agents view other agents' behavior as if it were time-invariant and underestimate the extent to which it becomes more difficult to recruit new members over time.

In equilibrium, the agents' beliefs about the other agents' behavior are statistically correct, and can be interpreted as a result of learning from partial feedback about the behavior in similar games that were played in the past (e.g., similar schemes organized by the SO). One motivation for the agents' coarse reasoning is that obtaining feedback about the aggregate behavior in these past schemes' induced games might be easier than gathering information about the time and context in which each offer was made.

Formally, for each $i \in I$ we denote by $H_i^1$ the set of information sets in which $i$ chooses whether or not to purchase a license, and by $H_i^2$ the set of information sets in which $i$ chooses whether or not to make an offer. Let $M_1 := \cup_{i \in I} H_i^1$ and $M_2 := \cup_{i \in I} H_i^2$. We refer to $M_1$ and $M_2$ as the agents' analogy classes and denote by $r_\sigma(h)$ the objective probability of reaching $h \in M_1 \cup M_2$ conditional on the profile $\sigma$ being played. For each $i \in I$, $\beta^i = (\beta_1^i, \beta_2^i)$ are agent $i$'s analogy-based expectations about the other agents' behavior. A strategy $\sigma_i$ is a best response to $\beta^i$ if it is optimal given a belief that each agent $j \neq i$ accepts every offer he receives with probability $\beta_1^i$ and that, if $j$ has the opportunity to make an offer, then he makes it with probability $\beta_2^i$. Let $\beta := (\beta^i)_{i \in I}$.

**Definition 2** *Agent i's analogy-based expectations $\beta^i$ are said to be consistent with the profile of strategies $\sigma$ if, for every $k \in \{1, 2\}$, it holds that $\beta_k^i = \frac{\sum_{h \in M_k} r_\sigma(h)\sigma(h)}{\sum_{h \in M_k} r_\sigma(h)}$ whenever $r_\sigma(h) > 0$ for some $h \in M_k$.*

**Definition 3** *The pair $(\sigma, \beta)$ forms an analogy-based expectation equilibrium (ABEE) if, for each $i \in I$, $\beta^i$ is consistent with $\sigma$ and $\sigma_i$ is a best response to $\beta^i$.*

Consistency implies that, in an ABEE, $\beta_1^i = \beta_1^j$ and $\beta_2^i = \beta_2^j$ for every pair of agents $i, j \in I$. Therefore, without loss of generality, we shall omit the superscript and use $\beta_1$ and $\beta_2$ instead.

*Discussion: Consistency, analogy classes, and the SO's strategy*

*Consistency.* Definition 1 does not place any restrictions on the agents' beliefs about analogy classes that are not reached with strictly positive probability. We can refrain from placing such restrictions as the only equilibria in which $M_1$ and $M_2$ are not reached with strictly positive probability are equilibria in which the SO never makes any offers, which are of secondary interest and do not change our results.

Consistency implies that the agents' expectations $\beta_1$ match the proportion of accepted offers. An important feature of consistency is that information sets are weighted according to the likelihood of their being reached. To see this, let $n = 3$ and consider a profile $\sigma$ in which the SO makes an offer in period 1 and, in each period $t \in \{2, 3\}$, every $i \in D_{t-1}$ makes an offer if he meets an agent. Moreover, suppose that agent $i_1$ accepts the SO's offer and all other agents reject every offer they receive. Note that agents $i_1$ and $i_2$ always receive an offer under $\sigma$. Agent $i_3$ receives an offer with probability $\frac{2}{3}$ since, with probability $\frac{1}{3}$, he meets agent $i_2$ who does not have a license. Only the first of the $\frac{8}{3}$ offers is accepted. Hence, $\beta_1 = \frac{1}{1+1+\frac{2}{3}} = \frac{3}{8} > \frac{1}{3}$ is consistent with $\sigma$.

*Analogy classes.* Each agent $i$'s analogy classes, $M_1$ and $M_2$, consist of all of the information sets in which agents move, including information sets in which $i$ himself moves. This is consistent with interpreting $i$'s behavior as best responding to coarse feedback about the behavior in similar games that were played in the past by a different set of agents (i.e., $i$ himself did not play in these games). Note that since $i$ was not a player in these past games, his own actions do not affect his analogy-based expectations.

We could exclude the information sets in which agent $i$ moves from his own analogy classes. These alternative analogy classes are consistent with the interpretation of $i$'s behavior as best responding to coarse feedback about the behavior in similar games in which $i$ himself played in the past. Our results hold under both types of partitions.[16]

*The SO's strategy.* The solution concept does not require that the SO's strategy be optimal. Thus, effectively, the SO is allowed to commit to a strategy. He can potentially benefit from such commitment as his behavior affects $\beta$. The SO's commitment power allows us to simplify the exposition, but does not affect the results.

## 4.2 Structure and existence of pyramid scams

In this section, we show that the SO cannot sustain a pyramid scam by means of a scheme that pays the distributors only for the number of licenses they sell (Theorem 2). Thus, under an ABEE, behind every pyramid scam is a scheme that pays for at least two levels of downline recruits. Theorem 3 shows that if the number of potential participants is sufficiently large, a pyramid scam is sustainable and the necessary condition of Theorem 2 is tight. Finally, Theorem 4 shows that the SO may have to use a scheme that pays for more than two levels of downline recruits to sustain a pyramid scam. Combining these results establishes that increasing the number of levels increases profits of scams and can be essential to making them viable.

We start by showing that the SO cannot sustain a pyramid scam by means of a reward scheme that compensates distributors only for the number of agents they directly recruit. We shall refer to such schemes as 1-level schemes.

**Definition 4** *A reward scheme $R$ is said to be a $z$-level scheme if $a_\tau = 0$ and $b_\tau = 0$ for every $\tau > z$, and $a_z \neq 0$ or $b_z \neq 0$.*

**Theorem 2** *Let $q = 0$. There exists no IC 1-level scheme $R$ such that $\pi^{ABEE}(R) > 0$.*

[16] In fact, the alternative analogy classes were employed in an earlier version of this paper (Antler, 2018).

To gain intuition for this result, let us suppose for a moment that agents purchase licenses in an ABEE and study their analogy-based expectations. Since the likelihood of meeting new entrants decreases over time, there is a period $t$ such that agents accept offers up to period $t$ and reject offers afterward. Nonetheless, distributors continue making offers after period $t$ because they falsely believe that the other agents might accept them. In fact, each distributor tries to recruit every person he meets,[17] and so makes, in expectation, $v_t$ offers after period $t$, where $v_t$ is as given in (1). Thus, every offer that is accepted in periods $1, ..., t$ results in a distributor who makes, in expectation, $v_t$ offers after period $t$, which are all rejected. We can conclude that the proportion of accepted offers to total offers, $\beta_1$, cannot exceed $\frac{1}{1+v_t}$.

Imagine that agent $i_t$, the last agent who is supposed to purchase a license in our ABEE, is contemplating an offer. Conditional on accepting, he expects to make $v_t$ offers and falsely believes that each of them will be accepted with probability $\beta_1$. The more offers he expects to make, the more offers are made (and rejected) late in the game, which implies that $\beta_1$ is lower. Overall, $i_t$ falsely expects to recruit $v_t\beta_1$ agents, which is less than 1 regardless of how large $v_t$ is. If the scheme is IC, then $a_1 \leq \phi$, which means that no commission on his own sales would cover the license fee. Hence, an ABEE in which agents buy licenses cannot exist in a 1-level scheme's induced game.

The next result establishes that the SO can sustain a pyramid scam by means of a 2-level scheme if $n$ is large.

**Theorem 3** *Let $q = 0$. There exists an integer $n^\star$ such that an IC 2-level scheme $R$ for which $\pi^{ABEE}(R) > 0$ exists if and only if $n \geq n^\star$.*

The intuition for the existence result resembles the intuition behind the ABEE analysis of the finite-horizon centipede game (Jehiel, 2005) and the capped bubble game (Moinas and Pouget, 2013). In an ABEE in which agents accept offers, they accept offers up to some period $t$ and reject offers afterward. However, analogy-based reasoners view the other agents' behavior as if it were time-invariant: each agent falsely believes that each of the other agents accepts offers with probability $\beta_1$, even after period $t$. Thus, agent $i_t$ falsely expects that he will recruit new entrants, that his recruits will recruit new entrants, and so on. This overoptimistic belief is what makes agent $i_t$ pay the license fee and join the pyramid.

[17]If $a_1 \geq \hat{c}$, distributors always find it optimal to recruit the new entrant. Note that, in a 1-level scheme, if $a_1 < \hat{c}$, then purchasing a license leads to a negative profit no matter how many individuals one recruits, and so our conjectured ABEE cannot exist.

What is the difference between multilevel schemes and 1-level schemes? Multilevel and, in particular, 2-level schemes induce contracts that require prospective participants to assess not only the number of people they will recruit in the future but also the number of people their recruits will recruit in the future. The agents' imperfect perception of the other agents' behavior leads them to overestimate both of these variables. As we showed in Theorem 2, agents whose beliefs are statistically correct do not overestimate their own ability to recruit by much and, therefore, the SO cannot overcome the incentive-compatibility constraint and sustain a pyramid scam by means of a 1-level scheme. In a similar manner, the agents do not overestimate their recruits' ability to sell licenses by much. However, the accumulation of agents' mistakes allows the SO to overcome the incentive-compatibility constraint and sustain a pyramid scam.

In light of Theorem 3 it is elementary to ask whether there are cases in which the SO must use more than two levels of compensation. The next result provides an affirmative answer by showing that the required network size for exploiting the agents is smaller for 3-level schemes than it is for 2-level schemes.

**Theorem 4** *There exists an integer $n^{\star\star} < n^{\star}$ such that, for every $n \geq n^{\star\star}$, there exists an IC 3-level scheme $R$ such that $\pi^{ABEE}(R) > 0$.*

In conclusion, the results in this section show that if the network size is sufficiently large, then the SO can sustain a pyramid scam. Sustaining a scam requires a scheme that charges a license fee and pays for recruitment, as the good is not intrinsically valued. Moreover, it requires paying for at least two levels of downline recruits and, in some instances, for strictly more than two levels. These results are different from the case in which agents are fully rational, where the SO never finds it beneficial to charge license fees, pay recruitment commissions, or use schemes that pay for more than two levels of downline sales.

*Generality of the results: A road map*

Throughout this section, we imposed three main assumptions. First, we assumed that $q = 0$, which made it salient that trade is inefficient. Second, we assumed that agents' beliefs are statistically correct, which constrained the overoptimism of the agents and the SO's ability to exploit them. Finally, we used a simple network formation process, which allowed us to abstract from situations in which agents are connected to multiple agents upstream. We now relax the first two assumptions. In Section 5, we let $q > 0$

and show that, when the demand is low, the profit-maximizing schemes resemble the ones identified in this section, but when the demand is high, the profit-maximizing schemes resemble the ones we identified in the rational expectations model of Section 3. In Section 6, we adapt Brunnermeier and Parker's (2005) model of motivated beliefs to our framework and provide additional support to the finding that multiple levels of recruitment commissions and license fees are inherent in profit-maximizing pyramid scams. In the concluding section, we reexamine the results of this section under a slightly different network formation model. Moreover, in a previous version of the paper (Antler, 2021, pp. 43-50), the results of Sections 3 and 4 are reexamined under various alternative specifications for the social network model.

# 5 Multilevel Marketing of Genuine Goods ($q > 0$)

We now examine an environment where the good is intrinsically valued and the agents are vulnerable to deceptive practices. Thus, the SO can benefit both from the agents' sales and from their mistakes. We start by studying a setting in which pyramid scams are viable and the demand for the good is low (fixed $n > n^{\star}$, small $q > 0$). We show that profit-maximizing schemes charge license fees and pay for multiple levels of downline recruits, like the pure pyramid scams in Section 4. We then turn to the case where the demand is high (fixed $q > 0$, large $n$) and show that, as in the rational expectations case of Section 3, profit-maximizing schemes do not charge license fees and do not pay recruitment commissions. Corollary 2 then concludes that, for large values of $n$, if $q$ is large, the profit-maximizing scheme looks like a "legitimate scheme" whereas if $q$ is small it looks like a pyramid scam. After establishing these results, we study the implications of banning the use of "pyramidal" components, namely, license fees and recruitment commissions.

We impose two mild assumptions in this section. First, note that due to their risk neutrality and their different beliefs, both the agents and the SO can benefit from raising the stakes of the contract. To guarantee that a profit-maximizing scheme exists, we shall assume that the maximal amount that each agent can pay for a license is $B$ and that $B$ is large w.r.t. $c$ and $\hat{c}$. Second, to simplify the analysis, we fix an arbitrary integer $\tau^{\star} > 2$ and restrict attention to $\tau^{\star}$-level schemes; i.e., we impose that $a_{\tau} = b_{\tau} = 0$ for every $\tau > \tau^{\star}$.

We start the analysis by showing that when $q$ is sufficiently small, then profit-maximizing schemes charge a license fee and pay recruitment commissions.

**Proposition 2** *For every $n > n^\star$, there exists a number $q^\star(n) > 0$ such that if $q < q^\star(n)$, then every IC profit-maximizing scheme charges a license fee and pays for at least two levels of downline recruits.*

When $q$ goes to zero, the SO's potential sales revenue goes to zero as well. However, the potential profit from the agents' mistakes does not vanish: agents are still willing to pay for distribution licenses given the "right" scheme and a budget $B$ that is not too small. To maximize his expected profit, the SO takes advantage of this mistake, which, as noted in the previous section, requires charging a license fee and paying for at least two levels of downline recruits.

Next, we establish that when the potential gains from sales are sufficiently large, schemes that pay recruitment commissions or charge license fees are not profit-maximizing.

**Proposition 3** *Fix $q > 0$. There exists a number $\hat{n}(q)$ such that if $n > \hat{n}(q)$, then, in every IC profit-maximizing scheme, $a_1 = a_2 = ... = b_3 = b_4 = ... = \phi = 0$, $b_1 > 0$, and $b_2 > 0$.*

Charging a license fee enables the SO to make a profit from the agents' mistakes. However, as noted in Section 3, the license fee has an additional, indirect, negative effect on the SO's ability to make a profit from the agents' sales. A fee makes it more costly to become a distributor, which requires paying higher sales commissions to attract prospective distributors. This effect becomes stronger when the potential profit from sales $qn$ is large. In such instances, the SO uses many distributors to increase the number of agents who purchase the good. Thus, he has to pay multiple commissions for every sale and every recruitment such that raising the commissions to compensate for the license fee becomes extremely costly and the SO is better off not charging a license fee. Note that an IC scheme that does not charge a license fee cannot pay for recruitment either. Thus, profit-maximizing schemes do not charge license fees and do not pay recruitment commissions.

Proposition 3 shows that when the demand for the good is high, the SO cannot benefit from extending the scheme beyond level two. Note that this is different from the case in which the demand for the good is low, where the SO may benefit from extending the scheme beyond level two (Theorem 4 shows this for $q = 0$ and can be extended to the case of small $q > 0$).

The main findings of this section are summarized in the following corollary.

**Corollary 2** *There exists a number $\tilde{n}$ such that for every $n > \tilde{n}$:*

- *If $q$ is sufficiently large, then in every IC profit-maximizing scheme $a_1 = ... = a_{\tau^\star} = b_3 = ... = b_{\tau^\star} = \phi = 0$.*
- *If $q$ is sufficiently small, then in every IC profit-maximizing scheme $\phi > 0$, $a_1 > 0$, and $a_2 > 0$. Moreover, it is possible that $a_\tau > 0$ for $\tau > 2$.*

We can conclude that license fees and recruitment commissions are useful for the SO when the demand is low, but not when it is high. This result is related to Heidhues et al.'s (2017) findings in the context of hidden add-on prices. They show that firms tend to benefit from this deceptive practice only in markets in which socially invaluable products (i.e., with lower social surplus than an alternative) are traded. The reason for this effect in their setting is that, when a product is socially valuable, firms may be able to increase market share and profits by drawing the consumers' attention to the hidden fees and undercutting prices.

*Banning recruitment commissions and license fees*

A natural way to limit the SO's ability to exploit the distributors' naivety is to ban the use of license fees, recruitment commissions, and more than two levels of sales commissions. However, as Proposition 2 shows, there are instances in which such a regulation strictly reduces the SO's potential profit. The next result establishes a bound on this effect by showing that, even under such restrictions, the SO's potential expected payoff is no lower than his potential expected payoff when he is not restricted and agents are fully rational. The latter payoff can be viewed as the *fundamental value* of the operation. An implication of this result is that the proposed regulation would not cause a business that could survive in a world with fully rational participants to shut down.

**Proposition 4** *For every scheme $R$, there exists an IC 2-level scheme $\tilde{R}$ such that $\tilde{a}_1 = \tilde{a}_2 = ... = \tilde{b}_3 = \tilde{b}_4 = ... = \tilde{\phi} = 0$ and $\pi^{ABEE}(\tilde{R}) \geq \pi^{PBE}(R)$.*

In Section 3, it was established that when agents are fully rational there is an IC profit-maximizing 2-level scheme $\tilde{R}$ in which $\tilde{a}_1 = \tilde{a}_2 = ... = \tilde{b}_3 = \tilde{b}_4 = ... = \tilde{\phi} = 0$, i.e., $\pi^{PBE}(\tilde{R}) \geq \pi^{PBE}(R)$ for every IC scheme $R$. The proof of Proposition 4 shows that the agents' behavior in a PBE of $\Gamma(\tilde{R})$ is identical to their behavior in an ABEE of $\Gamma(\tilde{R})$, and so $\pi^{ABEE}(\tilde{R}) = \pi^{PBE}(\tilde{R})$.

To see why analogy-based reasoners behave as if they were fully rational in $\Gamma(\tilde{R})$, note that these agents correctly predict (i) how many units of the good they will sell in the future and (ii) how many units their recruits will sell *conditional on buying a license*. In the ABEE/PBE of $\Gamma(\tilde{R})$, these are the only variables agents have to estimate in order to assess whether to buy a license and whether to recruit new entrants.

*The social optimum*

The IC 2-level scheme $\hat{R}$ that was introduced in Section 3 incentivizes the same socially optimal behavior both in an ABEE and in a PBE. The intuition and proof for this result are similar to those of Proposition 4. For brevity, they are omitted.

**Corollary 3** *There exists an IC socially optimal scheme that does not charge a license fee, does not pay recruitment commissions, and does not use more than two levels of sales commissions.*

In general, IC socially optimal schemes are not profit-maximizing. When $qn$ is large, they induce more distributors and pay higher commissions than profit-maximizing schemes as in the baseline model of Section 3 and for similar reasons.[18] When $n$ is large but $q$ is small such that $qn$ is small, IC socially optimal schemes are not profit-maximizing either, but for a different reason: they induce fewer distributors than profit-maximizing schemes (whose main objective is to benefit from the fees these distributors pay). Indeed, Proposition 2 shows that even when $q$ is small such that MLM is not socially optimal, profit-maximizing schemes can incentivize some of the agents to purchase a license. Thus, under ABEE, compared to the social optimum, when the demand is low profit-maximizing schemes induce too many distributors but when it is high they induce too few distributors.

# 6 Alternative Behavioral Explanations

The main behavioral model used in this paper captures the idea that agents do not fully grasp how difficult it becomes to recruit new members over time. In this section, we consider an alternative model of distorted beliefs that captures a psychological phenomenon that appears relevant in the context of MLM and adapt it to our setting. The analysis of this model provides further support to the finding that the potential

[18] This is shown formally in a previous version of this paper (see, Antler, 2021, pp. 51-53)

profit of a scam is increasing in the number of levels and that the scope to overestimate the value of a distributorship is increasing in the number of levels. We then conclude this section with a general discussion about the type of behavioral biases that can help sustain scams, and the features that make them more profitable.

In this section, we use Brunnermeier and Parker's (2005) model of self-deception in which individuals derive anticipatory utility from expecting that good things will happen in the future. In their model, individuals deceive themselves into holding (potentially) false beliefs, while balancing between anticipatory utility gains and the fact that distorted beliefs can lead to suboptimal choices in the future.[19] Brunnermeier and Parker show that this type of reasoning leads to speculation in situations where returns are positively skewed, such as in MLM.[20]

We now adapt Brunnermeier and Parker's behavioral model to our framework. In the spirit of their prescription, we assume that "each agent's beliefs are set taking as given the reaction functions of other agents." We simplify the model to the bare minimum by setting $\hat{c} = 0$ such that every distributor who meets an agent finds it optimal to recruit him. This allows us to posit that distributors always make an offer when they meet an agent and focus only on the agents' decisions whether to buy a license or not. To guarantee that a profit-maximizing scheme exists, we assume as in Section 5 that agents cannot pay more than $B > 0$ for a license.

We modify the baseline model by assuming that the order in which agents enter the game is drawn uniformly at random at the beginning of the game and that agents do not know the time at which they enter. When an agent receives an offer he updates his beliefs based on the identity of the proposer. We denote by $PR_i(t|j)$ the objective probability of $i$ being the $t$-th entrant conditional on receiving an offer from $j \in I \cup \{SO\}$. On the equilibrium path, $PR_i(t|j)$ is determined according to Bayes' law.[21]

Agent $i$'s strategy $s_i : \{SO, 1, ..., n\} \to \{0, 1\} \times \Delta(\{1, ..., n\})$ specifies a belief $\mathbf{r_i} \in \Delta(\{1, .., n\})$ and a purchasing decision $\sigma_i \in \{0, 1\}$ as a function of the player who made an offer to $i$. We use $r_i(t|j)$ to denote $i$'s subjective belief that he is the $t$-th entrant conditional on receiving an offer from $j$. For each $i \in I$, let $s_{-i} = (s_j)_{j \in I - \{i\}}$ and denote agent $i$'s expected rewards given that the other agents play $s_{-i}$ and $i$

[19] In economics, the idea that individuals trade off mental-consumption and accurate-decision values of beliefs originated with Akerlof and Dickens (1982) and has been widely explored since then. For a comprehensive review of the economics literature on motivated beliefs see Bénabou and Tirole (2016).

[20] See, e.g., Herbalife (2019).

[21] We refrain from placing restrictions on the agents' beliefs off the equilibrium path as the equilibria we analyze have full support.

purchases a license in period $t$ from $j$ by $p_t(s_{-i}, j)$. Let $\psi \in (0, 1)$ be the weight that captures the magnitude of the agents' anticipatory utility. A profile of strategies forms an equilibrium if for each $i \in I$ and $j \in I \cup \{SO\}$ agent $i$'s strategy maximizes

$$\sigma_i(j)\left[(1-\psi)\sum_{t=1}^{n} PR_i(t|j)p_t(s_{-i}, j) + \psi\sum_{t=1}^{n} r_i(t|j)p_t(s_{-i}, j) - \phi - c\right], \tag{2}$$

subject to the requirements that (1) $r_i(t|j) = 0$ whenever $PR_i(t|j) = 0$ and (2) $\sigma_i(j)$ maximizes

$$\sigma_i(j)\left[\sum_{t=1}^{n} r_i(t|j)p_t(s_{-i}, j) - \phi - c\right]. \tag{3}$$

To simplify the analysis, we shall focus on *symmetric equilibria*, that is, equilibria in which $s_i(k) = s_j(k)$ for every $i, j \in I$ and $k \in I \cup \{SO\}$.

Proposition 5 establishes that the potential profit of a scam is strictly larger with multilevel schemes than it is with 1-level schemes. In particular, if the weight $\psi$ agents put on anticipatory utility is intermediate, then sustaining a pyramid scam requires using a multilevel scheme.

**Proposition 5** *Set $q = 0$. There exists $\tilde{n}$ such that for every $n > \tilde{n}$, there exist $\psi^\star(n) > 0$ and $\psi^{\star\star}(n) > \psi^\star(n)$ such that:*

- *If $\psi < \psi^\star(n)$, then there exists no IC scheme $R$ such that $\pi^{BP}(R) > 0$.*
- *If $\psi \in [\psi^\star(n), \psi^{\star\star}(n))$, then there exists no IC 1-level scheme $R$ such that $\pi^{BP}(R) > 0$, but there exists an IC multilevel scheme $R$ such that $\pi^{BP}(R) > 0$.*
- *If $\psi \geq \psi^{\star\star}(n)$, then there exists an IC 1-level scheme $R$ such that $\pi^{BP}(R) > 0$. However, every scheme that is both 1-level and IC is not profit-maximizing.*

To gain intuition for Proposition 5, consider an agent who buys a license. He maximizes anticipatory utility by deceiving himself into believing that he entered the game first (or second) and, hence, that he is likely to recruit many agents. Since the likelihood of meeting new entrants goes down over time, he expects to do most of the recruiting early in the game. Thus, he expects to recruit mostly early entrants who are also likely to recruit many new members themselves. Overall, our "first" entrant falsely expects that he will recruit $v_1$ agents and that these agents will, in turn, recruit

$\sum_{t=2}^{n-1} \frac{v_t}{t} > v_1$ agents.[22] We can conclude that our agent expects to recruit fewer agents than the total number of agents he expects his recruits to recruit.

The above argument implies that, all else equal, our agent is better off if the SO increases $a_2$ and reduces $a_1$ by the same amount. For the SO, such a change saves costs: he pays $a_1$ for every distributor $j$ such that $d(SO, j) > 1$ and pays $a_2$ only if $d(SO, j) > 2$. Thus, a 2-level scheme can induce a higher expected payoff for the SO and a higher perceived payoff for the agents. Hence, the SO can sustain a scam by means of a 2-level scheme when it is impossible to do so by means of a 1-level scheme (intermediate $\psi$), and can increase his expected profit by means of a 2-level scheme when it is possible to sustain a pyramid scam by means of a 1-level scheme (high $\psi$).

We have established that the extent to which an agent overestimates the value of a distributorship and the potential profit of a scam increase when the SO moves from a 1-level to a 2-level scheme. It is possible to apply the same argument to show that, for every $z$, if $n$ is sufficiently large, then there exists an IC $(z + 1)$-level scheme that induces a strictly higher expected profit for the SO than every IC $z$-level scheme. That is, the extent to which the SO can exploit the agents strictly increases with the number of levels.

We conclude that license fees and multiple levels of recruitment commissions are inherent in profit-maximizing scams when agents have motivated beliefs. This insight is similar to the one obtained in the ABEE framework; however, none of these tools is useful for a legitimate business that faces fully rational agents as in our baseline model of Section 3.

*Discussion: What type of behavioral biases can support a pyramid scam?*

In general, a variety of behavioral biases can support pyramid scams. However, not every bias can support such a scam. Any behavioral bias that can support a pyramid scam must be able to overcome the challenge that typically pyramid scams will collapse, and backward induction typically will unravel the pyramid from this point backwards, implying that pyramids never start. This suggests that a behavioral bias that leads to a strategic error is needed: the last agents to join the pyramid must not realize that they are the last to join.

We presented two examples of models that lead to such errors: the ABEE in which agents do not understand the time-contingent nature of the other agents' behavior,

[22] When $\psi < 1$ this effect is mitigated by the weight $1 - \psi$ the agent puts on his objective beliefs (but does not disappear).

and the model of self-deception in which agents have false beliefs about their location in the game tree. In these models and others, what causes agents to overestimate their number of recruits will also cause them to overestimate the recruitment success of their recruits, and of their recruits' recruits, and so on. As a result, the size of the mistake that can be exploited accumulates to a larger and more profitable sum as more levels are incorporated into the reward scheme. We conclude that the more levels the scheme has, the more profitable it is for the SO to exploit a given bias via a pyramid scam.

# 7 Concluding Remarks

Legitimate MLM and fraudulent pyramid scams are two widespread phenomena. We developed a model that promotes a better understanding of the compensation structures that underlie these phenomena and explains why MLM compensation structures appear to be correlated with scams. The paper shows that a company can make a profit even in instances in which participants' beliefs are statistically correct and its product has no intrinsic value. Sustaining such a scam requires the company to charge a license fee and pay for at least two levels of downline recruits. We illustrated that when agents hold distorted overoptimistic beliefs that are not statistically correct, it might be possible to sustain such a pyramid scam by means of a 1-level scheme; however, maximizing the company's profit from a scam requires using a multilevel scheme. In all of these pyramid scams, the company lures the distributors by paying recruitment commissions and makes its profit from the fees they pay. The paper's benchmark results show that companies with a "good" product that face rational agents find it detrimental to use these two tools.

In the model, MLM enables companies that produce good products to incentivize agents to sell the product and recruit others to the sales force, thereby reaching pools of customers who would not purchase the product otherwise. It is natural to ask what advantage MLM brings to such companies compared to more traditional marketing methods. One answer is that MLM can be appealing in situations where there is a great deal of uncertainty about quality or fit. Consumers who are skeptical about a product's quality or whether it fits their needs would be reluctant to buy it at a high price. Traditional marketing channels may not allow the company to credibly communicate the merits of its product. However, when a consumer buys from a friend as in the MLM model, the latter might be better informed about the consumer's specific needs and the social capital in the interaction may allow him to credibly explain the merits of the product. Hence, MLM can outperform traditional marketing by decreasing uncertainty, thereby increasing the consumer's willingness to pay for the product.

It is possible to embed the baseline model in a simple framework in which the SO first learns $q$ and then chooses whether to use MLM as in the baseline model or a more traditional marketing channel. The parameter $q$ can be interpreted as the share of the population for whom the product fits. Ex ante, each agent $i$ does not know whether the product is suitable for his needs ($\omega_i = 1$) or not ($\omega_i = 0$). As assumed throughout this paper, in the MLM model, when agent $i$ meets a distributor he becomes aware of the product and learns $\omega_i$. Thus, with probability $q$ the agent is willing to pay 1 for the good. Under "traditional marketing," an agent $i$ who is targeted by the SO becomes aware of the product but does not learn $\omega_i$. Thus, the agent is willing to pay $q$ for the product with probability 1. Instead of commissions, the SO incurs a marketing cost of $m$ per agent he targets. Thus, his profit if he targets $n$ agents is $n(q - m)$. In conclusion, for small values of $q$, the SO is better off using MLM, while for large $q$, the SO might be better off using a traditional marketing channel.

We shall conclude by discussing a few modifications of the baseline setting.

*Incentive compatibility*

Throughout the paper we assumed that the SO uses only IC schemes to prevent distributors from manipulating him by creating fictitious players. The incentive-compatibility constraint prevents these manipulations when the SO can verify the identity of any distributor who wishes to be paid (in practice, to be paid, MLM distributors are often required to identify themselves). An SO who cannot verify the distributors' identities may wish to use a reward scheme where $\sum_{\tau=1}^{n} a_\tau \leq \phi$ and $\sum_{\tau=1}^{n} b_\tau \leq \eta$ in order to prevent each distributor from creating a tree of fictitious recruits and collecting the commissions that all the nodes in the tree would be eligible for.

Below, we extend the network formation model and show that while 1-level schemes cannot sustain a pyramid scam, 2-level schemes can sustain one, even under the stronger incentive-compatibility constraint. An additional illustration of this effect is provided in Section 6, where we consider a model of motivated beliefs.[23] The common component in these cases is that there are agents who overestimate the number of downline distributors they will have more than they overestimate the number of people they will recruit themselves. In an ABEE, this occurs only in social networks where there

[23]To see this, note that the 2-level scheme $R'$ used in the proof of Proposition 5 satisfies $a'_1 + a'_2 \leq \phi'$.

are *some* agents who have fewer successors/friends than their successors/friends have, which is not the case under the uniform random recursive tree model.

*Extension: Recruiting one's friends*

An individual who joins a scheme may find it natural to first approach his immediate friends as approaching strangers is perhaps more difficult. Such individuals exhaust their best opportunities to recruit new members soon after they join. In order to roughly approximate this, we modify the network formation model such that each player can meet new entrants only in the first period after he enters the game (i.e., an agent who enters the game in period $t$ can meet new entrants only in period $t+1$), and we interpret these entrants as the agent's friends. Moreover, we assume that in each period $t \in \{1, ..., n\}$, $\mu_t > 0$ new agents enter the game such that agents who enter the game in period $t$ have, in expectation, $\mu_{t+1}/\mu_t$ friends they can make offers to. As in the baseline model, the agent who meets each entrant is drawn by nature uniformly at random. Observe that in this network formation model, unlike in the random recursive tree model, *agents do not necessarily expect to meet more agents than their successors.*

We now show that, as in the baseline model of Section 4.2, *1-level schemes cannot sustain a pyramid scam* if $a_1 \leq \phi$. Subsequently, we shall show that 2-level schemes can sustain such a scam when $a_1 + a_2 \leq \phi$.

Set $q = 0$ and consider a profile in which every agent who receives an offer up to period $k < n$ accepts it, every agent who receives an offer in period $k+1$ rejects it, and every distributor who meets an agent in periods $1, ..., k+1$ makes him an offer. Note that no offer is made after period $k+1$. Every symmetric ABEE in which the SO's expected profit is strictly positive has this threshold structure under this network formation process. Under this profile, $\mu_1 + ... + \mu_{k+1}$ offers are made and $\mu_1 + ... + \mu_k$ of them are accepted. Hence, $\beta_1 = \frac{\mu_1 + ... + \mu_k}{\mu_1 + ... + \mu_{k+1}}$ and $\beta_2 = 1$ are consistent with this profile. Let $\mu_z/\mu_{z-1} = min\,\{\mu_2/\mu_1, ..., \mu_{k+1}/\mu_k\}$ and consider an agent $i$ who purchases a license in period $z-1$. He analogy-based expects to sell $\frac{\mu_z}{\mu_{z-1}}\beta_1 \leq \frac{\mu_2 + ... + \mu_{k+1}}{\mu_1 + ... + \mu_{k+1}} < 1$ licenses, which, if $R$ is an IC 1-level scheme, will not cover the cost of becoming a distributor. Hence, it is impossible to sustain a pyramid scam by means of a 1-level scheme.

The profile that we have described above can be part of an ABEE of a 2-level scheme's induced game. For example, suppose that there are three periods, $k = 1$, $\mu_1 = 1$, $\mu_2 = 4$, and $\mu_3 = 40$. As we calculated above, $\beta_1 = 0.2$. An agent who enters the game in period 1 analogy-based expects to recruit $\beta_1\mu_2 = 0.8$ distributors and analogy-based expects that these distributors will recruit $\beta_1^2 \frac{\mu_3}{\mu_1} = 1.6$ distributors. If $R$ pays $a_1 = a_2 = 0.5\phi$, an agent who enters the game in period 1 analogy-based expects a payoff of $1.2\phi - \phi - c - 0.8\hat{c}$. An agent who purchases a license in period 2 analogy-based expects to sell $\beta_1 \frac{\mu_3}{\mu_2} = 2$ licenses and, therefore, he analogy-based expects a payoff of $-c - 2\hat{c}$ and thus he finds it optimal not to purchase a license. Hence, for large $\phi$, we have an IC 2-level scheme $R$ (that satisfies $a_1 + a_2 \leq \phi$), where $\pi^{ABEE}(R) > 0$.

*Time-contingent compensation*

Throughout the analysis we focused on time-invariant schemes in line with real-world MLM. Such schemes are relatively simple to explain to potential participants and may conceal the non-stationary nature of the environment from these potential participants. Moreover, they create fewer incentive problems than non-stationary schemes (that may incentivize agents to postpone joining the sales force).

A natural question that arises is whether the stationarity of the scheme limits the SO's expected payoff. In a previous version of this paper (Antler, 2021), it is shown that the answer is affirmative. First, when agents are fully rational the SO maximizes profits by charging a license fee that *decreases over time*. The decreasing fee enables the SO to perfectly discriminate the distributors on the basis of the time at which they purchase a license. This, in turn, allows him to maximize the social surplus and capture all of it. Second, when agents are analogy-based reasoners, the SO can benefit from charging a license fee that *increases over time*. The increasing fee loosens the incentive-compatibility constraint: when the fee increases over time, the commissions can be higher than the fee in some of the periods and so an agent who purchases a license does not need to recruit many new members to recoup the fee paid. As a result, the SO may be able to sustain a pyramid scam using a 1-level scheme with an increasing fee even in instances in which it would be impossible for him to sustain such a scam using a single IC multilevel scheme. Thus, the direction in which the fee evolves over time may provide some evidence as to whether the SO believes that he faces boundedly rational agents or not.

# References

[1] Abreu, D. and Brunnermeier, M. K. (2003): "Bubbles and Crashes," *Econometrica,* 71, 173–204.

[2] Akerlof, G. A., and Dickens, W. T. (1982): "The Economic Consequences of Cognitive Dissonance," *American Economic Review,* 72, 307–319.

[3] Antler, Y. (2018): "Multilevel Marketing: Pyramid-Shaped Schemes or Exploitative Scams," Working Paper 13054, CEPR.

[4] Antler, Y. (2021): "Multilevel Marketing: Pyramid-Shaped Schemes or Exploitative Scams," Working Paper, Retrieved from yairantler.com.

[5] Babaioff, M., Dobzinski, S., Oren, S., and Zohar, A. (2012): "On Bitcoin and Red Balloons." In *Proceedings of the 13th ACM Conference on Electronic Commerce,* pp. 56–73.

[6] Bénabou, R. and Tirole, J. (2016): "Mindful Economics: The Production, Consumption, and Value of Beliefs," *Journal of Economic Perspectives,* 30, 141–164.

[7] Bhattacharya, P. and Gastwirth, J. (1984): "Two Probability Models of Pyramid or Chain Letter Schemes Demonstrating the Unreliability of the Claims of Promoters," *Operations Research,* 32, 527–536.

[8] Bianchi, M. and Jehiel, P. (2010): "Bubbles and Crashes with Partially Sophisticated Investors," Mimeo.

[9] Bort, R. (2016): "John Oliver Says Multilevel Marketing Companies Like Herbalife Are Pyramid Schemes," *Newsweek*, November 7.

[10] Brunnermeier, M. K. and Parker, J. A. (2005): "Optimal Expectations, *American Economic Review,* 95, 1092–1118.

[11] Crawford, V. P., Kugler, T., Neeman, Z., and Pauzner, A. (2009): "Behaviorally Optimal Auction Design: Examples and Observations," *Journal of the European Economic Association*, 7, 377–387.

[12] Direct Selling Association (2014): "U.S. Direct Selling Data." Retrieved from https://www.dsa.org/docs/default-source/research/researchfactsheet2007-2014.pdf?sfvrsn=9e7ecea5_0.

[13] Direct Selling Association (2016): "Direct Selling in 2016: An Overview." Retrieved from http://www.dsa.org/benefits/research/factsheets.

[14] Drmota, M. (2009): *Random Trees,* Vienna, Springer.

[15] Eliaz, K. and Spiegler, R. (2006): "Contracting with Diversely Naive Agents," *Review of Economic Studies,* 73, 689–714.

[16] Eliaz, K. and Spiegler, R. (2008): "Consumer Optimism and Price Discrimination," *Theoretical Economics*, 3, 459–497.

[17] Emek, Y., Karidi, R., Tennenholtz, M., and Zohar, A. (2011): "Mechanisms for Multi-Level Marketing." In *Proceedings of the 12th ACM Conference on Electronic Commerce*, pp. 209–218.

[18] Eyster, E. and Rabin, M. (2005): "Cursed Equilibrium," *Econometrica*, 73, 1623–1672.

[19] *FTC v. Fortune Hi-Tech Marketing, Inc.* 13 CV 578 (N.D. Ill, 2013).

[20] Federal Trade Commission (2016a): "Herbalife Will Restructure Its Multi-level Marketing Operations and Pay 200 Million for Consumer Redress to Settle FTC Charges." Retrieved from https://www.ftc.gov.

[21] Federal Trade Commission (2016b): "Multilevel Marketing." Retrieved from https://www.ftc.gov /tips-advice/business-center/guidance/multilevel-marketing.

[22] Gabaix, X. and Laibson, D. (2006): "Shrouded Attributes, Consumer Myopia, and Information Suppression in Competitive Markets," *Quarterly Journal of Economics,* 121, 505–540.

[23] Gastwirth, J. (1977): "A Probability Model of a Pyramid Scheme," *American Statistician,* 31, 79–82.

[24] Grubb, M. D. (2009): "Selling to Overconfident Consumers," *American Economic Review,* 99, 1770–1807.

[25] Grubb, M. D. (2015): "Overconfident Consumers in the Marketplace," *Journal of Economic Perspectives,* 29, 9–36.

[26] Heidhues, P. and Kőszegi, B. (2010): "Exploiting Naïvete about Self-Control in the Credit Market," *American Economic Review,* 100, 2279–2303.

[27] Heidhues, P. and Kőszegi, B. (2018): "Behavioral Industrial Organization." In *Handbook of Behavioral Economics*, 1, 6, 517–612.

[28] Heidhues, P., Kőszegi, B., and Murooka, T. (2017): "Inferior Products and Profitable Deception," *Review of Economic Studies,* 84, 323–356.

[29] Herbalife (2019): "U.S. Statement of Average Gross Compensation." Retrieved from https://edge.myherbalife.com/vmba/media/7B5C662A-D82A-4C1C-AA42-3F21238AAF54/Web/General/Original/StatementOfAverageGrossCompensation.pdf.

[30] Jehiel, P. (2005): "Analogy-Based Expectation Equilibrium," *Journal of Economic Theory*, 123, 81–104.

[31] Jehiel, P. (2011): "Manipulative Auction Design," *Theoretical Economics*, 6, 185–217.

[32] Jehiel, P. and Koessler, F. (2008): "Revisiting Games of Incomplete Information with Analogy-Based Expectations," *Games and Economic Behavior*, 62, 533–557.

[33] Keep, W. and Vander Nat, P. (2002): "Marketing Fraud: An Approach for Differentiating Multilevel Marketing from Pyramid Schemes," *Journal of Public Policy & Marketing*, 21, 139–151.

[34] Keep, W. and Vander Nat, P. (2014): "Multilevel Marketing and Pyramid Schemes in the United States. An Historical Analysis," *Journal of Historical Research*, 6, 188–210.

[35] Koehn, D. (2001): "Ethical Issues Connected with Multilevel Marketing," *Journal of Business Ethics*, 29, 153–160.

[36] Laibson, D. and Yariv, L. (2007): "Safety in Markets: An Impossibility Theorem for Dutch Books," Working Paper 867, Society for Economic Dynamics.

[37] Mahmoud, H. (1991): "Limiting Distributions for Path Lengths," *Probability in the Engineering and Informational Sciences,* 5, 53–59.

[38] Moinas, S. and Pouget, S. (2013): "The Bubble Game: An Experimental Study of Speculation," *Econometrica,* 81, 1507–1539.

[39] Piccione, M. and Rubinstein, A. (2003): "Modeling the Economic Interaction of Agents with Diverse Abilities to Recognize Equilibrium Patterns," *Journal of the European Economic Association,* 1, 212–223.

[40] Rantala, V. (2019): "How Do Investment Ideas Spread through Social Interaction? Evidence from a Ponzi Scheme," *Journal of Finance,* 74, 2349–2389.

[41] SEC (2013): "Beware of Pyramid Schemes Posing as Multi-Level Marketing Programs." Retrieved from https://www.sec.gov/oiea/investor-alerts-bulletins/investor-alerts-ia_pyramidhtm.html.

[42] Shiller, R. J. (2015): *Irrational Exuberance,* Princeton, Princeton University Press.

[43] Spiegler, R. (2011): *Bounded Rationality and Industrial Organization,* New York, Oxford University Press.

[44] Stivers, A., Smith, D., and Jin G. Z. (2019): "The Alchemy of a Pyramid: Transmutating Business Opportunity into a Negative Sum Wealth Transfer," SSRN Working Paper.

[45] Tirole, J. (1982): "On the Possibility of Speculation under Rational Expectations," *Econometrica*, 50, 1163–1182.

[46] World Federation of Direct Selling Associations (2018): "Annual Report 2017/2018." Retrieved from https://www.wfdsa.org.

# Appendix: Proofs

**Proof of Lemma 1**. In expectation, regardless of the players' strategies, an agent who purchases a license in period $t$ meets $qv_t$ agents who are willing to pay $1 \geq \eta$ for the good. Because of this independence, in a PBE, both on and off the equilibrium path, all agents correctly expect that, conditional on purchasing a license, agent $i_t$ will sell $qv_t$ units of the good.

**Part 1.** For sufficiency, note that selling $qv_t$ units yields a payoff of $qv_tb_1 - c - \phi$ to agent $i_t$ even if the latter does not sell licenses. For necessity, let $t$ be the last period in which an agent buys a license in a PBE (on and off the equilibrium path). Agent $i_t$ cannot expect to sell licenses in this PBE. Thus, he can expect a payoff of at most $qv_tb_1 - c - \phi$ conditional on purchasing a license. Sequential rationality implies that $qv_tb_1 - c - \phi \geq 0$. Since $v_t$ is decreasing in $t$, it holds that $b_1qv_k \geq c + \phi$ for every $k \leq t$.

**Part 2.** For sufficiency, note that a distributor who recruits agent $i_t$ earns $a_1$ for the recruitment and $b_2$ for every unit $i_t$ sells. Thus, if $a_1 + b_2qv_t \geq \hat{c}$ the distributor finds it optimal to recruit $i_t$ even if he does not expect $i_t$ to sell licenses. For necessity, consider a PBE and let $t$ be the last period in which both (i) $qv_tb_1 \geq c + \phi$ and (ii) a distributor is supposed to make an offer to an agent (on and off the equilibrium path). Since $qv_tb_1 \geq c + \phi$, agent $i_t$ accepts the offer. Moreover, the distributor cannot expect $i_t$ to recruit anyone and, therefore, the distributor believes that making an offer to $i_t$ yields an expected payoff of $a_1 + b_2qv_t - \hat{c}$. Sequential rationality implies that $a_1 + b_2qv_t - \hat{c} \geq 0$. Since $v_t$ is decreasing in $t$, $a_1 + qb_2v_k \geq \hat{c}$ holds for every $k \leq t$.

**Proof of Theorem 1**. Let $R$ be an IC scheme that charges $\phi > 0$ and $\eta \leq 1$, and denote by $\sigma$ a profile of strategies that is part of a PBE in which the SO's expected profit is $\pi^{PBE}(R)$. Denote $k_1 = sup\{t|b_1qv_t \geq \phi + c\}$. By Lemma 1, if $k_1 < 2$ under $\sigma$, then at most one agent purchases a license in every PBE of $\Gamma(R)$, and so $R$ is not profit-maximizing. In the remainder of the proof, we assume that $k_1 \geq 2$. If $a_1 + b_2 q v_1 < \hat{c}$, let $k_2 = 1$ and, otherwise, let $k_2 = sup\{t | a_1 + b_2 q v_t \geq \hat{c} \ and \ q v_t \geq \phi + c\}$. Since $v_t$ is decreasing in $t$, Lemma 1 implies that every offer made up to (resp., after) period $k_1$ under $\sigma$ is accepted (resp., rejected) and every distributor who meets an agent up to (resp., after) period $k_2$ recruits (resp., does not recruit) the latter.

To show that $R$ is not profit-maximizing, consider a scheme $R'$ such that $\eta' = \eta$, $a'_1 = \phi' = 0$, $b'_1 q v_{k_1} = c$, $b'_2 q v_{k_2} = \hat{c}$, and $a_2 = a_3 = ... = b_3 = b_4 = ... = 0$. Since $v_t$ is decreasing in $t$, Lemma 1 implies that the agents' behavior in $\sigma$ is part of a PBE of $\Gamma(R')$. Hence, to show that $\pi^{PBE}(R') > \pi^{PBE}(R)$ we only need to show that the SO's expected payoff when $\sigma$ is played is greater in $\Gamma(R')$ than in $\Gamma(R)$.

Since $\eta = \eta'$, given $\sigma$, the SO's revenue from sales is the same under both schemes. We now show that the expected net transfers (i.e., after subtracting the license fees) from the SO to the distributors when $\sigma$ is played are greater in $\Gamma(R)$.

Consider the expected net transfers from the SO to a distributor $i_t$. If $d(SO, i_t) = 1$ (i.e., the SO recruits $i_t$), then, under $R$, the SO collects $\phi$ from $i_t$ and pays him $b_1$ for each of his sales. Under $R'$, the SO pays $i_t$ $b'_1$ for each sale. Note that

$$-\phi + q v_t b_1 \geq -\phi + q v_t (\frac{c + \phi}{q v_{k_1}}) \geq q v_t \frac{c}{q v_{k_1}} = q v_t b'_1, \tag{4}$$

with the second inequality being weak for $t = k_1$ and strict for $t < k_1$.

If $d(SO, i_t) > 1$ (i.e., $i_t$ is recruited by a distributor $j \in I$), then, under $R$, the SO collects $\phi$ from $i_t$, pays $a_1$ to $j$, and, for each of $i_t$'s sales, the SO pays $b_1$ to $i_t$ and $b_2$ to agent $j$. Under $R'$, the SO pays $b'_1$ to $i_t$ and $b'_2$ to agent $j$ for each retail sale made by $i_t$. The SO's expected net transfers to the distributors are greater under $R$ if

$$a_1 - \phi + q v_t (b_1 + b_2) \geq a_1 (1 - \frac{v_t}{v_{k_2}}) - \phi (1 - \frac{v_t}{v_{k_1}}) + \frac{v_t \hat{c}}{v_{k_2}} + \frac{v_t c}{v_{k_1}} \geq q v_t (b'_1 + b'_2). \tag{5}$$

Since $R$ is IC, $a_1 \leq \phi$. As $t \leq k_2 \leq k_1$, (5) holds. We conclude that for every $t \leq k_1$ (resp., $t < k_1$) the expected net transfers based on $i_t$'s recruitment and sales are weakly (resp., strictly) lower in $R'$ under $\sigma$. Hence, $\pi^{PBE}(R') > \pi^{PBE}(R)$.

Note that $b_1 \leq \eta'$ since $R$ is IC and that $b'_1 \leq b_1$. If $b'_2 > \eta$, then, in $\Gamma(R')$, the SO incurs a loss of $b'_1 + b'_2 - \eta$ whenever a sale is made by a distributor $i$ such that $d(SO, i) > 1$. The SO can earn more than $\pi^{PBE}(R')$ by using a scheme $R''$ that is identical to $R'$ except that $b''_2 = 0$ as distributors never recruit in $\Gamma(R'')$. Clearly, $R''$ is

IC. Thus, if $R'$ is not IC, then $R''$ is IC and $\pi^{PBE}(R'') > \pi^{PBE}(R)$.

We can conclude that $R$ is not profit-maximizing. This means that a profit-maximizing scheme must charge $\eta > 1$ (which induces a payoff of 0) or $\phi = 0$. Incentive compatibility implies that if $\phi = 0$, then $a_1 = 0$ as well.

**Proof of Proposition 1**. We prove this result by backward induction. Let $R$ be an IC scheme and consider a PBE of $\Gamma(R)$. Sequential rationality implies that agent $i_n$ rejects every offer he receives, both on and off the equilibrium path. Let $t \in \{1, ..., n-1\}$ and suppose that all agents reject every offer they receive (both on and off the path) after period $t$ in our PBE. Agent $i_t$ cannot expect to sell licenses. Since $q = 0$, he cannot expect to meet any agent who will buy the good at a price $\eta > 0$ regardless of the other agents' strategies. Being sequentially rational, he rejects every offer he receives both on and off the equilibrium path. By induction, no agent purchases a license in our PBE and, since $q = 0$, the SO does not sell the good. Thus, $\pi^{PBE}(R) = 0$.

**Proof of Theorem 2**. Consider an IC 1-level scheme $R$. If $a_1 \leq \hat{c}$, then, conditional on purchasing a license, an agent makes an expected payoff of at most $-c - \phi < 0$. Hence, no agent purchases a license in an ABEE of $\Gamma(R)$.

Suppose that $a_1 > \hat{c}$ and consider an ABEE of $\Gamma(R)$ in which the SO makes offers. Clearly, agent $i_n$ rejects every offer in this ABEE. Let $t \in \{1, ..., n-1\}$ and suppose that agents reject every offer they receive in periods $t+1, ..., n$. Since $a_1 \geq \hat{c}$, every agent who holds a license at the end of period $t$ makes, in expectation, $v_t$ offers in periods $t+1, ..., n$. Thus, for every offer that is accepted in periods $1, ..., t$ there are, in expectation, $v_t$ rejected offers in periods $t+1, ..., n$, and so the proportion of accepted offers, $\beta_1$, cannot exceed $\frac{1}{1+v_t}$. Hence, conditional on accepting an offer, agent $i_t$ falsely expects a payoff of $\beta_1 v_t(a_1 - \hat{c}) - c - \phi$. Since $R$ is IC, $a_1 \leq \phi$, and so $i_t$ rejects every offer he receives in our ABEE. We can conclude that no agent purchases a license in our ABEE. Since $q = 0$, no agent ever buys the good. Hence, $\pi^{ABEE}(R) = 0$.

**Proof of Theorem 3**. The proof consists of three parts. Part 1 shows that if $\pi^{ABEE}(R) > 0$ and $R$ is an IC 2-level scheme, then $a_1 \geq \hat{c}$. Part 2 shows that if there exists an IC 2-level scheme $R$ such that $\pi^{ABEE}(R) > 0$ when there are $n$ agents, then there exists an IC 2-level scheme $R'$ such that $\pi^{ABEE}(R') > 0$ when there are $n' > n$ agents. Part 3 establishes that such a scheme exists if $n$ is sufficiently large.

**Part 1.** Assume to the contrary that agents purchase licenses in an ABEE of an IC 2-level scheme $R$ in which $a_1 < \hat{c}$. A distributor who recruits agent $i_t$ expects to increase his payoff by $a_1 - \hat{c} + \beta_1\beta_2 v_t a_2$. Since $v_t$ is decreasing in $t$, there is a cutoff $k_2$ such that distributors make offers to every agent they meet up to period $k_2$ and make no offers afterward. Moreover, no agent ever purchases a license in any period $t \geq k_2$ as he knows that he will refrain from making offers. It follows that an agent who purchases a license in period $t < k_2$ falsely expects a payoff of

$$\sum_{j=t+1}^{k_2} \frac{1}{j}\beta_1[(a_1 - \hat{c}) + \beta_1\beta_2 v_j a_2] - c - \phi \leq \beta_1\beta_2 v_t(v_t - v_{k_2})\beta_1 a_2 - c - \phi. \tag{6}$$

Consider the last period $t$ in which an agent accepts an offer in our ABEE. Each offer accepted in periods $1, ..., t$ leads in expectation to (i) $v_t - v_{k_2}$ rejected offers after period $t$ and (ii) $v_t$ opportunities to make offers where $v_{k_2}$ of them are not made. Thus, $\beta_1 \leq \frac{1}{1+v_t-v_{k_2}}$ and $\beta_2 \leq \frac{1+v_t-v_{k_2}}{1+v_t}$. As $R$ is IC, $a_2 \leq \phi$. We conclude that (6) is strictly negative, which violates the optimality of purchasing a license in this ABEE.

**Part 2.** Consider an IC 2-level scheme $R$ such that $\pi^{ABEE}(R) > 0$. By Part 1, $a_1 \geq \hat{c}$ such that in an ABEE, every distributor makes an offer to every agent he meets. In an ABEE that maximizes the SO's expected payoff, there is a period $k$ such that $i_t$ accepts (resp., rejects) every offer he receives if $t \leq k$ (resp., if $t > k$) and the SO makes no offers after period $k$ (as such offers are rejected and lower the agents' expectations). In this ABEE, $\beta_1 = \frac{1}{1+v_k}$ and $\beta_2 = 1$. Hence, agent $i_k$ falsely expects a payoff of

$$\frac{v_k}{1+v_k}(a_1 - \hat{c}) + \frac{\sum_{j=k+1}^{n-1} \frac{v_j}{j}}{(1+v_k)^2} a_2 - c - \phi \geq 0, \tag{7}$$

where $\frac{v_k}{1+v_k}$ is the number of agents he expects to recruit and $\frac{\sum_{j=k+1}^{n-1} \frac{v_j}{j}}{(1+v_k)^2}$ is the number of agents he expects his recruits to recruit.

Note that (7) is increasing in $n$. Thus, for any $n' > n$, there is a scheme $R^{n'}$ that is identical to $R$ except that $\phi^{n'} > \phi$ such that a profile of strategies in which agents accept every offer up to period $k$, reject every offer made after period $k$, and in which the SO makes no offers after period $k$, is part of an ABEE of $\Gamma(R^{n'})$. If $\pi^{ABEE}(R) > 0$ when there are $n$ agents, then $\pi^{ABEE}(R^{n'}) > 0$ when there are $n'$ agents.

**Part 3.** Consider a profile of strategies $\sigma$ in which agent $i_1$ accepts an offer if he receives one and all other agents reject every offer they receive, the SO makes an offer only in period 1, and, conditional on purchasing a license, every agent makes an offer to every agent whom he meets. The SO's expected payoff is $\phi - \hat{c}$ under $\sigma$. Agent $i_1$'s analogy-based expected payoff in the ABEE that corresponds to $\sigma$ is given in the LHS of (7) for $k = 1$. As the harmonic sum diverges, when $n$ goes to infinity, (7) goes to $a_1 - \hat{c} + 0.5a_2 - c - \phi$. Thus, for a sufficiently large $n$, we can choose $\phi$, $a_1 \in (\hat{c}, \phi]$, $a_2 \leq \phi$ and $b_1 = b_2 = 0$ such that (7) holds in equality and the profile $\sigma$ is part of an ABEE in which the SO's expected payoff is strictly positive.

**Proof of Theorem 4**. Consider an IC 2-level scheme $R$. In part 2 of the proof of Theorem 3 we showed that if $\pi^{ABEE}(R) > 0$, then (7) must hold for some $k \leq n$. For $a_1 \leq \phi$ and $a_2 \leq \phi$, (7) is smaller than $\frac{v_k}{1+v_k}\phi + \sum_{j=k+1}^{n} \frac{v_j}{j}\phi - \phi - c$, which is strictly negative if $\phi \geq 0$, $n \leq 25$, and $k \leq n$.

We now show that for $n \geq 25$ there exists an IC 3-level scheme $R$ such that $\pi^{ABEE}(R) > 0$. Fix a profile $\sigma$ as described in part 3 of the proof of Theorem 3 and recall that $\sigma$ is consistent with $\beta_1 = \frac{1}{1+v_1}$ and $\beta_2 = 1$ and induces an expected payoff of $\phi - \hat{c}$ to the SO. Consider a 3-level scheme $R$ in which $a_1 = a_2 = a_3 = x\phi$. Under $\sigma$, the first entrant obtains an analogy-based expected payoff of

$$\beta_1 \sum_{j=2}^{n} \frac{(x\phi - \hat{c})}{j} + \beta_1^2\beta_2 \sum_{j=2}^{n-1} \sum_{j'=j+1}^{n} \frac{x\phi}{jj'} + \beta_1^3\beta_2^2 \sum_{j=2}^{n-2} \sum_{j'=j+1}^{n-1} \sum_{j''=j'+1}^{n} \frac{x\phi}{jj'j''} - \phi - c \tag{8}$$

and, conditional on purchasing a license, every agent who enters the game after period 1 obtains less than (8). For a sufficiently large $n$ (in particular, for $n \geq 25$), it is possible to choose a large $\phi$ and an $x < 1$ such that (8) equals 0, in which case, $\sigma$ is part of an ABEE of $\Gamma(R)$ in which the SO makes a strictly positive expected payoff.

**Proof of Proposition 2**. In Theorem 3, it was shown that for $n > n^\star$ and $q = 0$, there exists an IC reward scheme $R$ such that $\pi^{ABEE}(R) = \phi - \hat{c} >> 0$. The fact that $q > 0$ does not change the optimality of the agents' strategies w.r.t. their analogy-based expectations in $\Gamma(R)$ if we set $b_1 = b_2 = 0$. Thus, as long as $B \geq \phi$, $\phi - \hat{c}$ is a lower bound for the SO's expected payoff in a profit-maximizing scheme.

Consider a scheme $R'$ such that $\phi' = 0$. Note that the SO's expected payoff in an ABEE of $\Gamma(R')$ cannot exceed $qn$, which is his expected revenue in an ABEE in which all agents purchase licenses. Clearly, for a sufficiently small $q$, $qn < \phi - \hat{c}$.

Consider an IC scheme $\tilde{R}$ such that $\tilde{\phi} > 0$, $\tilde{a}_1 \geq 0$, and $a_2 = a_3 = ... = 0$, and an ABEE $(\sigma, \beta)$ of $\Gamma(\tilde{R})$. If agents do not purchase licenses in this ABEE, then, again, the SO's expected payoff cannot exceed $qn$. Assume that agents do purchase licenses and denote by $k_1$ the last period in which an agent does so.

If $\tilde{a}_1 \geq \hat{c}$, then, conditional on purchasing a license, every agent makes an offer to every agent he meets. Hence, as shown in the previous proofs, $\beta_1 \leq \frac{1}{1+v_{k_1}}$. Since $\tilde{R}$ is IC, agent $i_{k_1}$'s analogy-based expected payoff in this case cannot exceed

$$\tilde{\phi}\frac{v_{k_1}}{1+v_{k_1}} + q\frac{n-k_1}{k_1+1} - \tilde{\phi} - c, \tag{9}$$

where $\frac{n-k_1}{k_1+1}$ is the expected number of agents in the subtree of $G$ rooted at $i_{k_1}$. If $q$ is sufficiently small, then (9) is strictly negative, in contradiction to the optimality of $i_{k_1}$'s strategy.

To complete the proof, let $a_1 < \hat{c}$. Conditional on purchasing a license, agent $i_{k+1}$'s expected payoff cannot exceed $q\frac{n-k_1}{k_1+1} - \tilde{\phi} - c < 0$, in contradiction to the existence of an ABEE in which he purchases a license if $q$ is sufficiently small. In conclusion, for a sufficiently small $q$, the SO's expected payoff given a scheme that charges $\phi = 0$ and/or pays $a_2 = a_3 = ... = 0$ is smaller than $\phi - \hat{c}$, the lower bound for his expected payoff in a profit-maximizing scheme.

**Proof of Proposition 3**. Let $(R^n)_{n=1}^{\infty}$ be a sequence of IC schemes such that each $R^n$ is profit-maximizing when there are $n$ agents. For each $n \in \mathbb{N}$, let $(\sigma^n, \beta^n)$ be an ABEE of $\Gamma(R^n)$ that induces an expected profit of $\pi^{ABEE}(R^n)$, where $\beta^n = (\beta_1^n, \beta_2^n)$. We use $k_1^n$ to denote the last period in which the agents accept offers and $k_2^n$ to denote the last period in which distributors make offers in $\sigma^n$. Without loss of generality, we restrict attention to schemes that charge $\eta^n = 1$.

Throughout the proof, we use several technical results on random trees and perform several related calculations, all of which can be found in the Supplemental Appendix.

**Step 1: Lower bounds.** There exists a constant $\gamma < 1$ and an integer $n'$ such that, for every $n > n'$, it holds that $\pi^{ABEE}(R^n) > \gamma n$, $k_1^n > \gamma n$, and $k_2^n > \gamma n$. This step is proven in Technical Lemmata 2 and 3 in the Supplemental Appendix.

**Step 2: The dual problem.** The profit-maximizing scheme must minimize the SO's expected net cost among the class of IC schemes that charge $\phi^n$ and in which $\sigma^n$ is part of an ABEE of their induced game. This problem can be written as

$$\begin{aligned} & min_{a_1,...,a_{\tau^\star},b_1,...,b_{\tau^\star}} \textstyle\sum_{\tau=1}^{\tau^\star}(a_\tau\kappa(a_\tau)+b_\tau\kappa(b_\tau)) \\ s.t.\ (i_{k_1}) \quad & \textstyle\sum_{\tau=1}^{\tau^\star}[a_\tau w(a_\tau)+b_\tau w(b_\tau)] - \hat{c}w(a_1) \geq c+\phi^n \\ (i_{k_2}) \quad & \textstyle\sum_{\tau=1}^{\tau^\star}[a_\tau \hat{w}(a_\tau)+b_\tau \hat{w}(b_\tau)] \geq \hat{c} \\ (IC) \quad & a_1, a_2, ... \leq \phi^n \ and \ b_1, b_2, ... \leq 1, \end{aligned} \tag{10}$$

where $w(z)$ is the marginal increase in $i_{k_1}$'s willingness to pay for a license due to the commission $z \in \{a_1, b_1, ..., a_{\tau^\star}, b_{\tau^\star}\}$, $\hat{w}(z)$ is the corresponding increase in the distributors' benefit from recruiting agent $i_{k_2}$, and $\kappa(z)$ is the increase in the SO's cost that is associated with $z$.

**The SO's costs.** The SO pays $a_\tau$ for every distributor $j$ such that $d(SO,j) > \tau$. The expected number of such distributors is

$$\kappa(a_\tau) = \sum_{G' \in \mathcal{G}^n} \sum_{j \in G'} \frac{\mathbb{1}\left(d\left(SO,j\right) > \tau\right)}{min\{k_1^n, k_2^n\}!}, \tag{11}$$

where $\mathcal{G}^n$ is the set of rooted trees with $min\{k_1^n+1, k_2^n+1\}$ nodes, and $\mathbb{1}\left(d\left(SO,j\right) > \tau\right) \in \{0,1\}$ is an indicator that equals 1 if and only if $d\left(SO,j\right) > \tau$. The SO pays $b_\tau$ for every sale made by a distributor $j$ such that $d(SO,j) \geq \tau$. Thus, $\kappa(b_\tau) = q\kappa(a_\tau) + q\kappa(a_{\tau-1})v_{min\{k_1^n,k_2^n\}}$ for $\tau > 1$.

**Constraint $\mathbf{i_{k_1}}$.** The LHS is agent $i_{k_1^n}$'s willingness to pay for a license. If $k_1^n \geq k_2^n$, then $w(b_1) = qv_{k_1^n}$ and $w(z) = 0$ for any other commission $z \in \{a_1, ..., a_{\tau^\star}, b_2, ..., b_{\tau^\star}\}$ as $i_{k_1^n}$ does not expect to make offers in equilibrium. If $k_1^n < k_2^n$, then, for $\tau > 1$, it holds that $w(a_\tau) = (\beta_1^n)^\tau(\beta_2^n)^{\tau-1}\sum_{j=k_1^n+1}^{k_2^n}\frac{l_{j,\tau-1}}{j}$ and $w(b_\tau) = q(\beta_1^n)^{\tau-1}(\beta_2^n)^{\tau-2}\sum_{j=k_1^n+1}^{k_2^n}\frac{l_{j,\tau-1}}{j}$, where $l_{j,\tau}$ is the expected number of agents in the $\tau$-th level of the subtree of $G$ rooted at the $j$-th entrant. For $\tau = 1$, $w(b_1) = qv_{k_1}$ and $w(a_1) = \beta_1^n\sum_{j=k_1^n+1}^{k_2^n}\frac{1}{j}$. Note that $\hat{c}w(a_1)$ is $i_{k_1^n}$'s expected cost of training new recruits.

**Constraint $\mathbf{i_{k_2}}$.** The LHS is the increase in the expected reward of a distributor who recruits $i_{k_2}$. Clearly, $\hat{w}(b_1) = 0$ and $\hat{w}(a_1) = 1$. If $\tau > 1$, then $\hat{w}(a_\tau) = (\beta_1\beta_2)^{\tau-1}l_{k_2^n,\tau-1}$ and $\hat{w}(b_\tau) = q(\beta_1\beta_2)^{\tau-2}l_{k_2^n,\tau-1}$.

**Step 3: The profit-maximizing scheme is a 2-level scheme.** In the Supplemental Appendix (see inequalities (18), (19), and (20)) we use results on random trees to show that, if $n$ is sufficiently large, then, for any $z \in \{a_1, b_2\}$ and $z' \in \{a_3, b_3, ..., a_{\tau^\star}, b_{\tau^\star}\}$, it holds that $\frac{w(z)}{\kappa(z)} \geq \frac{w(z')}{\kappa(z')}$ and $\frac{\hat{w}(z)}{\kappa(z)} \geq \frac{\hat{w}(z')}{\kappa(z')}$ with at least one strict inequality. Because of the linearity of (10) in $w$, $\hat{w}$, and $\kappa$, it follows that if $z' > 0$ and $z' \in \{a_3, b_3, ..., a_{\tau^\star}, b_{\tau^\star}\}$, then $a_1 = \phi$ and $b_2 = 1$. If this is the case, then the SO's expected profit cannot exceed $\sum_{t=1}^{n} \frac{B+1}{t} + \sum_{t=1}^{n-1} \sum_{j=t+1}^{n} \frac{1}{tj}$, as the SO earns (at most) $1 + B$ from every agent he meets and 1 from every agent who meets these agents. The latter expression is smaller than $\gamma n$ for a sufficiently large $n$, in contradiction to the result obtained in Step 1.

**Step 4: For a sufficiently large $n$, it holds that $\phi = 0$ at the optimum**. Consider an IC scheme $R$ that charges $\phi > 0$. There are two cases to consider, (1) $a_1 \geq c$ and (2) $a_1 < c$. Consider the first case and note that $a_1 \geq c$ implies that a distributor who meets an agent finds it optimal to make an offer to the latter. Thus, $k_2^n = n$. As a result, $\hat{w}(a_2) = \hat{w}(a_1) = \hat{w}(b_1) = 0 < \hat{w}(a_1) = 1$. As shown in the Supplemental Appendix (inequality (18)), $\frac{w(a_1)}{\kappa(a_1)} > \frac{w(b_1)}{\kappa(b_1)} > \frac{w(a_2)}{\kappa(a_2)} > \frac{w(b_2)}{\kappa(b_2)}$. Hence, if $a_2 > 0$ or $b_2 > 0$, then $a_1 = \phi$ and $b_1 = 1$, which implies that the SO's expected profit cannot exceed $\sum_{t=1}^{n} \frac{B+q}{t}$ (i.e., the expected revenue from his sales and recruitments). For a sufficiently large $n$, the latter expression is smaller than the lower bound $\gamma n$ established in Step 1. It follows that if our scheme is profit-maximizing, then $a_2 = b_2 = 0$.

At the optimum, agent $i_{k_1}$ must be indifferent whether to purcahse a license as, otherwise, it would be possible to increase $\phi$ without changing the agents' equilibrium behavior. Thus, $b_1 q v_{k_1} + (a_1 - \hat{c})w(a_1) = \phi + c$, or $b_1 = \frac{\phi + c - (a_1 - \hat{c})w(a_1)}{q v_{k_1}}$.

We now show that $R$ is not profit-maximizing by introducing a different scheme, $R'$, such that $\pi^{ABEE}(R') > \pi^{ABEE}(R)$. Let $R'$ be a 2-level scheme such that $a_1' = a_2' = \phi' = 0$, $b_1' = \frac{c}{v_{k_1^n}}$, and $b_2' = \frac{\hat{c}}{q v_{k_1^n}}$. It is possible to verify that there exists an ABEE of $\Gamma(R)$ in which agents (resp., distributors) accept (resp., make) offers in periods $1, ..., k_1^n$ and reject (resp., do not make) offers in periods $k_2^n + 1, ..., n$. The SO's revenue from the agents' sales in this ABEE is identical to his revenue from the agents' sales in the ABEE of $\Gamma(R)$ as the same agents purchase licenses in both ABEEs. We now show that the expected net transfers (i.e., including the license fee) from the SO to the agents under $R$ are higher than under $R'$.

In the transition from $R$ to $R'$ the SO's profit is lowered by $\phi k_1^n$ since there is no fee under $R'$. The change from $b_1$ to $b_1'$ increases the SO's expected payoff by $\frac{\phi - (a_1 - \hat{c})w(a_1)}{q v_{k_1^n}} \kappa(b_1)$, the reduction in $a_1$ increases the SO's expected payoff by $a_1 \kappa(a_1)$, and the addition to $b_2$ decreases the SO's expected payoff by $\frac{\hat{c}}{q v_{k_1^n}} \kappa(b_2)$. Overall, the SO's expected payoff increases in the transition from $R$ to $R'$ if

$$-\phi k_1^n + \frac{\phi - (a_1 - \hat{c})w(a_1)}{q v_{k_1^n}} \kappa(b_1) + a_1 \kappa(a_1) - \frac{\hat{c}}{q v_{k_1^n}} \kappa(b_2) > 0.$$

Plugging $\kappa(b_1), \kappa(b_2)$, and $w(a_1)$ and manipulating yields

$$\frac{\phi\kappa(a_1)}{v_{k_1^n}} - (a_1 - \hat{c})\frac{(\kappa(a_1) + v_{k_1^n}k_1^n)}{1 + v_{k_1^n}} + (a_1 - \hat{c})\kappa(a_1) - \frac{\hat{c}\kappa(a_2)}{v_{k_1^n}} > 0.$$

By incentive compatibility, $\phi \geq a_1$ and, by assumption, $a_1 \geq \hat{c}$. By definition, $\kappa(a_1) \geq \kappa(a_2)$. Thus, the above inequality holds if

$$\frac{\kappa(a_1) - (v_{k_1^n})^2 k_1^n + (1 + v_{k_1^n})v_{k_1^n}\kappa(a_1)}{v_{k_1^n}(1 + v_{k_1^n})} > 0$$

Since $k_1^n > \gamma n$ for large $n$, it follows that $v_{k_1^n}$ is bounded from above for all $n$. Note that $k_1^n - \kappa(a_1)$ is equal to the $k_1^n$-th harmonic number, $h_{k_1^n}$, as it is equal to the expected number of nodes in first level of a random recursive tree. Thus, for a sufficiently large $n$ the above inequality holds, and so $R$ is not profit-maximizing. We can conclude that a scheme is profit-maximizing and IC only if it charges $\phi = 0$. Incentive compatibility implies that $a_1 = a_2 = ... = 0$ as well.

The proof for the case of $a_1 < \hat{c}$ is similar (for any IC scheme that charges $\phi > 0$, the transition to the scheme $R'$ increases the SO's expected payoff) and, therefore, it is omitted.

**Proof of Proposition 4**. Assume that agents are fully rational. As we showed in the proof of Theorem 1, there are three possible cases. In the first, there exists an IC profit-maximizing 2-level scheme $\tilde{R}$ in which $\tilde{\eta} = 1$, $\tilde{a}_1 = \tilde{a}_2 = \tilde{\phi} = 0$, $\tilde{b}_1 = \frac{c}{qv_{k_1}}$, and $\tilde{b}_2 = \frac{\hat{c}}{v_{k_2}}$, where $k_1 \in \{1, ..., n\}$ and $k_2 \in \{1, ..., k_1\}$. In the second, there exists an IC profit-maximizing 1-level scheme $\tilde{R}$ in which $\tilde{\eta} = 1$, $\tilde{a}_1 = \tilde{\phi} = 0$, $\tilde{b}_1 = \frac{c}{qv_{k_1}}$, and $k_1 \in \{1, ..., n\}$. In the third, the SO does not recruit at the optimum and the proof is immediate.

Consider the first case. By Lemma 1, in a PBE of $\Gamma(\tilde{R})$, every agent who receives an offer up to period $k_1$ accepts it and every agent who receives an offer afterward rejects it. Moreover, every distributor who meets an agent up to period $k_2$ makes an offer to the latter and every distributor who meets an agent after period $k_2$ does not make him an offer. We now show that this behavior is part of an ABEE of $\Gamma(\tilde{R})$, which implies that $\pi^{ABEE}(\tilde{R}) \geq \pi^{PBE}(R)$ for every IC scheme $R$.

In an ABEE of $\Gamma(\tilde{R})$, a distributor who recruits agent $i_t$ increases his (analogy-based) expected payoff by $qv_t\tilde{b}_2 - \hat{c}$. Hence, every distributor who meets an agent up to period $k_2$ makes an offer to the latter and every distributor who meets an agent after period $k_2$ does not make him an offer. As a result, an agent who purchases a license in period $t \geq k_2$ does not expect to make offers, and so, conditional on purchasing a license, he analogy-based expects a payoff of $qv_t\tilde{b}_1 - c$. Hence, $i_{k_1}$ is indifferent whether to purchase a license or not. Since $v_t$ is decreasing in $t$, agents who enter the game after $i_{k_1}$ reject every offer in our ABEE, and, because it is possible to refrain from making offers, agents who enter the game prior to $k_1$ accept every offer.

The proof of the second case follows the same logic and is omitted for brevity.

**Proof of Proposition 5**. In a symmetric equilibrium, either all of the agents accept every offer or none of the agents accept any offer. To see this, note that an agent $i$ accepts an offer only if there is at least one agent $j \in I - \{i\}$ such that $\sigma_j(i) = 1$. By symmetry, $\sigma_j(i) = 1$ implies that $\sigma_k(i) = 1$ for every $k \in I - \{i, j\}$. Finally, since the likelihood of meeting new entrants goes down over time, if agents accept every offer they receive from other agents, $p_t(s_{-i}, j)$ is decreasing in $t$ for every $i \in I$ and $j \in I \cup \{SO\}$. As a result, $\sum_{t=1}^{n} PR_i(t|SO)p_t(s_{-i}, SO) > \sum_{t=1}^{n} PR_i(t|j)p_t(s_{-i}, j)$ for every $i \in I$ and $j \in I - \{i\}$. Hence, $\sigma_j(i) = 1$ implies that $\sigma_j(SO) = 1$.

In an equilibrium $(s_i)_{i \in I}$ in which agents accept offers, it must be that $r_i(2|j) = 1$ and $r_i(1|SO) = 1$ for every $i \in I$ and $j \in I - \{i\}$. To see this, note that since $p_t(s_{-i}, j)$ is decreasing in $t$, these beliefs maximize (2) conditional on purchasing a license and satisfy the requirement that $r_i(t|j) = 0$ whenever $PR_i(t|j) = 0$. Furthermore, under these beliefs, (3) is greater than (2) such that the corresponding requirement is satisfied. We can conclude that an equilibrium in which agents accept offers exists only if (2) is positive for every $i \in I$ and $j \in I \cup \{SO\}$ given subjective beliefs $r_i(2|j) = 1$ for $j \in I$ and $r_i(1|SO) = 1$. Since $p_t(s_{-i}, j)$ is decreasing in $t$, an equilibrium in which agents accept offers exists if and only if (2) is positive when an agent receives an offer from another agent $j \in I$ and all agents accept every offer, i.e.,

$$(1 - \psi) \sum_{t=1}^{n} PR_i(t|j)p_t(s_{-i}, j) + \psi p_2(s_{-i}, j) - \phi - c \geq 0. \tag{12}$$

Consider an IC 1-level scheme $R$. If all of the other agents accept every offer, $p_t(s_{-i}, j) = v_t a_1$. Thus, $\pi^{BP}(R) > 0$ if and only if

$$(1 - \psi) \sum_{t=1}^{n} \frac{t - 1}{t(n - 1 - v_1)} v_t a_1 + \psi v_2 a_1 - \phi - c \geq 0. \tag{13}$$

Since the harmonic sum diverges and (13) is continuous and increasing in $\psi$, if $n$ is sufficiently large, then there is a cutoff $\overline{\psi} < 1$ such that an equilibrium in which agents accept offers exists if and only if $\psi \geq \overline{\psi}$. Since $a_1 \leq \phi$ in every IC scheme, it follows that a cutoff $\psi^{\star\star}(n)$ exists.

We now show that if $n$ is sufficiently large and $\psi \geq \psi^{\star\star}(n)$, then 1-level schemes are not profit-maximizing. Let $R$ be an IC 1-level scheme such that $\pi^{BP}(R) > 0$. Let $R'$ be a 2-level scheme in which $\phi' = \phi$ and $a'_1 = a'_2 = 0.5a_1$. Under both $R'$ and $R$, when an agent $j$ is recruited the SO pays $a_1$ if $d(SO, j) > 2$. If $d(SO, j) = 2$, then under $R'$, the SO pays $0.5a_1$, while under $R$ the SO pays $a_1$. Hence, if there is an equilibrium of $\Gamma(R')$ in which agents purchase licenses, then $\pi^{BP}(R') > \pi^{BP}(R)$. Such an equilibrium exists if and only if

$$\frac{(1-\psi)a_1}{2}\sum_{t=1}^{n-1}\frac{t-1}{t(n-1-v_1)}(v_t + \sum_{k=t+1}^{n}\frac{v_k}{k}) + \frac{\psi a_1}{2}[v_2 + \sum_{t=3}^{n-1}\frac{v_t}{t}] - \phi - c \geq 0. \qquad (14)$$

If $a_1 \leq \phi$ and (13) is positive, then (14) is strictly greater than (13). This has two implications. First, 1-level schemes are never profit-maximizing. Second, since (14) is continuous in $\psi$, there exists $\psi < \psi^{\star\star}(n)$ such that the SO can sustain a pyramid scam by means of a 2-level scheme. The existence of a cutoff $\psi^{\star}(n) < \psi^{\star\star}(n)$ such that the SO can sustain a pyramid scam if and only if $\phi \geq \psi^{\star}(n)$ follows from (12) being increasing and continuous in $\psi$ given optimal beliefs.

# Supplemental Appendix: Technical results and Calculations

*Technical results: Linear bounds for Step 1 of Proposition 3*

**Lemma 2** *Fix $\gamma' > 0$ such that $q(log\,(1/\gamma')\,(1 - \frac{c+\hat{c}}{q(log(1/\gamma')-1)}) - \hat{c} > \gamma'$. There exists a number $n(\gamma')$ such that for every $n > n(\gamma')$ it holds that $\pi^{ABEE}(R^n) \geq {\gamma'}^2 n$.*

**Proof.** Denote $k = \lceil \gamma' n \rceil$. Let $R$ be a 1-level scheme in which $\eta = 1$, $\phi = \hat{c}$, $a_1 = \hat{c}$, and $b_1 = \frac{c+\hat{c}}{qv_k}$. Consider a profile $\sigma$ in which every agent accepts (resp., rejects) every offer he receives up to (resp., after) period $k$ and every distributor makes an offer to every agent he meets. It is easy to verify that $\sigma$ is part of an ABEE of $\Gamma(R)$. For a sufficiently large $n$, it holds that $1 + v_k > log\,(1/\gamma')$ and $b_1 < \frac{c+\hat{c}}{q(log(1/\gamma')-1)}$. The SO's expected payoff under $\sigma$ is greater than $k\Big(q[1+v_k]\,(1-b_1) - \hat{c}\Big)$. Hence, for a sufficiently large $n$, it holds that

$$\pi^{ABEE}\,(R^n) \geq \pi^{ABEE}\,(R) \geq \gamma' n \left( q(log\,(1/\gamma')\,(1 - \frac{c+\hat{c}}{q(log(1/\gamma')-1)}) - \hat{c} \right) \geq {\gamma'}^2 n. \quad (15)$$

■

**Lemma 3** *There exist $\bar{\gamma} \in (0,1)$ and $n\,(\bar{\gamma}) \in \mathbb{N}$ such that $min\{k_1^n, k_2^n\} > \bar{\gamma} n$ for every $n > n\,(\bar{\gamma})$.*

**Proof.** The SO obtains a revenue of at most $1 + B$ from every distributor and 1 from each agent who meets a distributor but does not purchase a license. Hence, in an ABEE in which $k_2 \geq k_1$, the SO obtains an expected revenue smaller than

$$k_1^n \left(1 + B + v_{k_1^n}\right) + v_{k_1^n} < k_1^n(2 + B + log\,(n/k_1^n)) + log\,(n) + 1.$$

In an ABEE in which $k_2 < k_1$, the SO obtains an expected revenue smaller than

$$k_2^n \left(1 + B + v_{k_2^n}\right) + \sum_{j=k_2^n+1}^{k_1} \frac{v_j + 1 + B}{j} + v_{k_1^n} < k_2^n(2 + B + log(n/k_2^n)) + (1 + B + log(n))^2.$$

There exists $\bar{\gamma} > 0$ that satisfies $\bar{\gamma} log\,(1/\bar{\gamma}) < min\,\{\gamma^2/2, 1/e\}$ and $n\,(\bar{\gamma})$ such that if $n > n\,(\bar{\gamma})$ and $min\{k_1^n, k_2^n\} \leq \bar{\gamma} n$, then the linear lower bound on the SO's expected profit in (15) is greater than both of the above upper bounds. ■

We can conclude that there exists a constant $\gamma < 1$ and an integer $n'$ such that, for every $n > n'$, it holds that $\pi^{ABEE}(R^n) > \gamma n$, $k_1^n > \gamma n$, and $k_2^n > \gamma n$. This proves Step 1 in the proof of Proposition 3.

*Technical results on random trees: Completion of Steps 3 and 4 in Proposition 3.*

**Lemma 4** *For every $j \in \{1, ..., n-1\}$ and $\tau > 1$ it holds that $v_j l_{j,\tau-1} \geq 2l_{j,\tau}$.*

**Proof.** Note that if $j + \tau > n$, then $l_{j,\tau} = 0$. Observe that $v_j l_{j,1} = 2l_{j,2} + \sum_{z=j+1}^{n} \frac{1}{z^2}$ if $j \leq n-1$. We prove the lemma by induction on the size of $\tau$. We assume that $v_j l_{j,\tau-1} \geq 2l_{j,\tau}$ and show that it implies that $v_j l_{j,\tau} \geq 2l_{j,\tau+1}$. We can write the latter inequality as:

$$v_j \left( \frac{l_{j+1,\tau-1}}{j+1} + \frac{l_{j+2,\tau-1}}{j+2} + ... \right) \geq 2 \left( \frac{l_{j+1,\tau}}{j+1} + \frac{l_{j+2,\tau}}{j+2} + ... \right). \quad (16)$$

We can combine the induction hypothesis with the fact that $v_j$ is weakly decreasing in $j$ to see that (16) holds. ■

*Calculating the "cost-benefit" ratios $\frac{w}{\kappa}$ and $\frac{\hat{w}}{\kappa}$ (Step 3)*

We need to show that, for a sufficiently large $n$ it holds that $\frac{w(z)}{\kappa(z)} \geq \frac{w(z')}{\kappa(z')}$ and $\frac{\hat{w}(z)}{\kappa(z)} \geq \frac{\hat{w}(z')}{\kappa(z')}$ with at least one strict inequality for every $z \in \{a_1, b_2\}$ and $z' \in \{a_3, ..., a_{\tau^\star}, b_3, ..., b_{\tau^\star}\}$.

First, consider the SO's expected cost $\kappa(a_\tau)$ that is given in (11) and note that this is essentially the expected number of the first $x = min\{k_1^n, k_2^n\}$ entrants whose distance from the SO is greater than $\tau$. The next lemma shows that when $x$ goes to infinity, the share of the first $x$ entrants whose distance from the SO is greater than $\tau$ goes to 1.

**Lemma 5** *For every $\tau \in \mathbb{N}$ it holds that*

$$lim_{n\to\infty} \sum_{G' \in \mathcal{G}^n} \sum_{j \in G} \frac{\mathbb{1}(d(SO, j) > \tau)}{xx!} = 1. \quad (17)$$

**Proof.** The tree $G$ is a uniform random recursive tree rooted at the SO, and the distance $d(SO, i_t)$ corresponds to the *insertion depth* of the $(t+1)$-th node. Theorem 1 in Mahmoud (1991) establishes that the normalized insertion depth $M_i^\star = \frac{M_i - log(i)}{\sqrt{log(i)}}$ has the limiting distribution $\mathcal{N}(0, 1)$, i.e., the standard normal distribution. Thus, the proportion of nodes inserted at a distance greater than $\tau$ from the root on the LHS of (17) goes to 1 when the size of the random tree, $x$, goes to infinity (to obtain the LHS of (17), note that the number of trees of size $x+1$ is $x!$). ■

By Lemma 3, $min\{k_1^n, k_2^n\} > \overline{\gamma}n$ for a sufficiently large $n$. Thus, Lemma 5 implies that $lim_{n\to\infty}\frac{\kappa(a_\tau)}{\kappa(a_{\tau+1})} = 1$ and $lim_{n\to\infty}\frac{\kappa(b_\tau)}{\kappa(b_{\tau+1})} = 1$ for every $\tau \in \{2, ..., \tau^\star - 1\}$. Moreover, $lim_{n\to\infty}\frac{\kappa(a_1)}{\kappa(a_2)} = 1$.

We now split the analysis into three cases: (1) $n = k_2^n > k_1^n$, (2) $n > k_2^n > k_1^n$, and (3) $k_2^n \leq k_1^n$.

**Case 1.** The analogy-based expectations in this case are $\beta_1^n = \frac{1}{1+v_{k_1^n}}$ and $\beta_2^n = 1$. Thus, $\frac{w(a_\tau)}{w(b_\tau)} = \frac{1}{q(1+v_{k_1^n})}$. Recall from the main text that $\kappa(b_\tau) = q\kappa(a_\tau) + q\kappa(a_{\tau-1})v_{k_1^n}$ for $\tau > 1$ and $\kappa(b_1) = q\kappa(a_1) + qk_1^n v_{k_1^n}$. By definition, $k_1^n > \kappa(a_1) > ... > \kappa(a_{\tau^\star})$. Moreover, by Lemma 5, $\frac{\kappa(a_\tau)}{k_1^n}$ goes to 1 when $n$ goes to infinity for every $\tau \in \{1, ..., \tau^\star\}$. Thus, for a sufficiently large $n$ it holds that $\frac{w(a_\tau)}{\kappa(a_\tau)} > \frac{w(b_\tau)}{\kappa(b_\tau)}$. By Lemma 4 it holds that $w(a_\tau) \geq 2w(a_{\tau+1})$ and $w(b_\tau) \geq 2w(b_{\tau+1})$. We can conclude that, for a sufficiently large $n$, it holds that

$$\frac{w(a_1)}{\kappa(a_1)} > \frac{w(b_1)}{\kappa(b_1)} > \frac{w(a_2)}{\kappa(a_2)} > \frac{w(b_2)}{\kappa(b_2)} > \frac{w(z)}{\kappa(z)} \tag{18}$$

for every $z \in \{a_3, ..., a_{\tau^\star}, b_3, ..., b_{\tau^\star}\}$. Note that $\hat{w}(a_1) = 1 > 0 = w(z)$ for every $z \in \{a_2, ..., a_{\tau^\star}, b_1, ..., b_{\tau^\star}\}$.

**Case 2.** The analogy-based expectations in this case are $\beta_1 = \frac{1}{1+v_{k_1^n} - v_{k_2^n}}$ and

$$\beta_2^n = \frac{k_1^n - \sum_{t=1}^{k_1^n}\frac{1}{t} + k_1^n(v_{k_1^n} - v_{k_2^n})}{k_1^n - \sum_{t=1}^{k_1^n}\frac{1}{t} + k_1^n v_{k_1^n}},$$

where $\sum_{t=1}^{k_1^n}\frac{1}{t}$ represents the SO's expected number of offers in periods $t = 1, ..., k_1^n$. Hence, $\beta_1^n\beta_2^n \leq \frac{1}{1+v_{k_1^n}}$. Since $\frac{k_1^n}{k_1^n - \sum_{t=1}^{k_1^n}\frac{1}{t}}$ goes to 1 when $n$ goes to infinity, for a sufficiently large $n$, $\beta_1^n\beta_2^n$ is arbitrarily close to $\frac{1}{1+v_{k_1^n}}$. Note that $\frac{q}{\beta_1^n\beta_2^n}w(a_\tau) = w(b_\tau)$ for $\tau > 1$. Thus, for a sufficiently large $n$, $\frac{w(a_\tau)}{\kappa(a_\tau)}$ is arbitrarily close to $\frac{w(b_\tau)}{\kappa(b_\tau)}$ for $\tau > 1$. As above, we can use Lemma 4 to show that $w(a_\tau) \geq 2w(a_{\tau+1})$ for $\tau \geq 1$ and $w(b_\tau) \geq 2w(b_{\tau+1})$ for $\tau > 1$. We can conclude that

$$\frac{w(a_1)}{\kappa(a_1)} > \frac{w(b_2)}{\kappa(b_2)} > \frac{w(z)}{\kappa(z)} \tag{19}$$

for every $z \in \{a_3, ..., a_{\tau^\star}, b_3, ..., b_{\tau^\star}\}$. We can repeat the above exercise for the ratio $\frac{\hat{w}}{\kappa}$ (the proof is identical to the one for the ratio $\frac{w}{\kappa}$ and, therefore, it is omitted).

**Case 3.** The analogy-based expectations in this case are $\beta_1^n = 1$ and

$$\beta_2^n = \frac{k_2^n - \sum_{t=1}^{k_2^n} \frac{1}{t}}{k_2^n - \sum_{t=1}^{k_2^n} \frac{1}{t} + k_2^n v_{k_2^n} + \sum_{j=k_2^n+1}^{k_1^n} \frac{v_j}{j}}.$$

It is easy to see that $w(b_1) > 0$ and $w(z) = 0$ for every $z \in \{a_1, ..., a_{\tau^\star}, b_2, ..., b_{\tau^\star}\}$, as agent $i_{k_1^n}$ does not expect to sell licenses. It is left to consider the ratio $\frac{\hat{w}}{\kappa}$. Clearly, $\hat{w}(a_1) = 1 > \frac{v_{k_2^n}}{1+v_{k_2^n}} \geq \beta_1^n \beta_2^n v_{k_2^n} = \hat{w}(a_2)$. Since $\beta_1^n \beta_2^n < \frac{1}{1+v_{k_2^n}}$, Lemma 4 implies that $\hat{w}(a_\tau) \geq 2\hat{w}(a_{\tau+1})$ for every $\tau > 1$. Similarly, Lemma 4 implies that $\hat{w}(b_\tau) \geq 2\hat{w}(b_{\tau+1})$ for every $\tau > 1$.

Note that Lemma 3 implies that $k_2^n > \bar{\gamma} n$ for large $n$. As a result, $v_{k_2^n}$ is bounded from above (e.g., by $log(\frac{1}{\bar{\gamma}})+1$). Furthermore, for a sufficiently large $n$, $\beta_2^n$ is arbitrarily close to $\frac{1}{1+v_{k_2^n}}$. This implies that, for a sufficiently large $n$, $\hat{w}(a_\tau)$ is arbitrarily close to $\hat{w}(b_\tau)$ for any $\tau > 1$. Moreover, since $v_{k_2^n}$ is bounded from above, $\hat{w}(a_2)$ is bounded below $1 = \hat{w}(a_1)$. We can conclude that, for a sufficiently large $n$, it holds that

$$\frac{\hat{w}(a_1)}{\kappa(a_1)} > \frac{\hat{w}(b_2)}{\kappa(b_2)} > \frac{\hat{w}(z)}{\kappa(z)} \tag{20}$$

for every $z \in \{a_3, ..., a_{\tau^\star}, b_3, ..., b_{\tau^\star}\}$.